Linear Algebra Preliminary Examination
April 2010

Problem 1. Suppose $A$ and $B$ are $n \times n$ matrices and $AB = 0$. Prove that $\text{rank } A + \text{rank } B \leq n$.

Problem 2. Suppose that $P$ and $Q$ are $n \times n$ matrices such that $P^2 = P$, $Q^2 = Q$, and $I - P - Q$ is invertible. Show that $P$ and $Q$ have the same rank.

Problem 3. Suppose $A$ and $B$ are orthogonal projections in a finite-dimensional complex inner product space $V$. Suppose for some vector $x \in V$ we have $ABx = x$. Prove that $Ax = Bx = x$.

Problem 4. Suppose $A$, $B$, and $C$ are $2 \times 2$ matrices. Suppose $AB = BA$, $AC = CA$, and $BC \neq CB$. Prove that $A$ is a scalar matrix.

Problem 5. Let $A$ and $B$ be complex $n \times n$ matrices. Prove or disprove:

(a): If $A$ and $B$ are diagonalizable, so is $AB$.

(b): If $A^2 = A$, then $A$ is diagonalizable.

(c): If $A$ is invertible and $A^2$ is diagonalizable, then $A$ is diagonalizable.

Problem 6. Suppose $A$ is a Hermitian matrix, and $B$ is some matrix. Suppose $A^3B = BA^3$. Prove that $AB = BA$. 

1