Ph.D. Preliminary Examination (Analysis)

April, 2010

INSTRUCTIONS: *Do all six problems. In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should contain the necessary details. All problems are worth the same number of points.*

1. Let I be the interval $[0, \infty)$. For $n \in \mathbf{N}$ and $t \in I$, let

$$f_n(t) = \sin((t + 4n^2\pi^2)^{1/2}).$$

(i) Show that the sequence $\{f_n\}$ is equicontinuous on I;
(ii) Show that $\{f_n\}$ does not contain a subsequence which is uniformly convergent on I.

2. Let $\sum_{n=1}^{\infty} u_n$ be a convergent series of real numbers. Use the definition of limit to prove that

$$\lim_{n\to\infty} \frac{u_1 + 2u_2 + \cdots + nu_n}{n} = 0.$$

3. Suppose that $f : \mathbf{R}^2 \to [0, \infty)$ is uniformly continuous on $\mathbf{R}^2$ and

$$\sup_{r>0} \left(\int\int_{x^2+y^2\leq r^2} f(x, y)dA \right) < \infty.$$

Prove that $\lim_{|(x,y)|\to\infty} f(x, y) = 0$.

4. Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x, y) = \begin{cases} \dfrac{x^2(x + y^2)}{x^2 + y^6} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Determine whether f is differentiable at $(0, 0)$ and prove your conclusion.

5. Let (M, d) be a metric space and S be a nonempty compact subset of M. Suppose that $F : S \to S$ satisfies

$$d(F(x), F(y)) < d(x, y)$$

for all $(x, y) \in (S \times S)\backslash\{(s, s) : s \in S\}$. Prove that there exists a unique ω in S such that $F(\omega) = \omega$.

6. Let $D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$ and let u be a nonconstant real-valued C^2 function on a neighborhood of D which satisfies $u(x, y) = 0$ for all $(x, y) \in \partial D$. Prove that

$$\int\int_D u\Delta u dA < 0.$$