Linear Algebra Preliminary Examination
April 2009

Problem 1.

Suppose A and B are similar matrices. Prove the following.

a) A and B have the same set of eigenvalues.

b) For each eigenvalue the algebraic multiplicities in A and B are the same.

c) For each eigenvalue the geometric multiplicities in A and B are the same.

NOTE: You are not allowed to use without proof the Jordan canonical form theorem.

Problem 2.

Suppose an $n \times n$ matrix A has n distinct positive real eigenvalues. Prove that there are exactly 2^n distinct matrices B such that $B^2 = A$.

Problem 3.

Let V be an n-dimensional linear vector space and $N : V \to V$ be a nilpotent map of index n, i.e., such that $N^n = 0$ and $N^{n-1} \neq 0$.

a) Show that the eigenvalues of N are zero.

b) Show that $(I - N)$ is invertible. Find $(I - N)^{-1}$.

c) Show that V has a basis consisting of $x, Nx, ..., N^{n-1}x$ where $x \in V$.

Problem 4.

Suppose A, B, C, D are $n \times n$ matrices such that $AC = BC$ and $AD = -BD$. Suppose that A is invertible. Prove that $\text{rank}(C) + \text{rank}(D) \leq n$.

Problem 5.

Suppose $v_1, v_2, \ldots, v_n$ are vectors in $\mathbb{R}^{n-1}$ such that $v_i \cdot v_j < 0$ for all $i \neq j$. Prove that there exist non-negative real numbers $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

Problem 6.

a) Show that if the self-adjoint part of a complex square matrix A is positive-definite, then A is invertible and the self-adjoint part of A^{-1} is positive-definite.

b) Let a be a fixed positive real number. Show that if a self-adjoint complex matrix A is positive-definite then $||W|| < 1$, where $W = (I - aA)(I + aA)^{-1}$ and $||.||$ is a matrix norm induced by the standard Euclidean norm.

Preliminary Exam in Advanced Calculus

April 2009

Write solutions to different problems on separate sheets. Use scrap paper first to solve problems and write final solutions in a legible form taking care of clarity of presentation.

Problem 1. Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a function $f : \Omega \to \mathbb{R}$ is *locally bounded* if for every $x \in \Omega$ there is a ball $B(x, \varepsilon) \subset \Omega$ such that f is bounded in $B(x, \varepsilon)$. Note that we do not assume continuity of f at any point.

(a) Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is locally bounded, then it is bounded in *every* open ball in $\mathbb{R}^n$.

(b) Give an example of a locally bounded function defined in the unit ball of $\mathbb{R}^n$, $f : B^n(0,1) \to \mathbb{R}$ which is not bounded.

Problem 2. Assume that $f : [0,1] \to [0,1]$ is a continuous function such that the set $\{x \in [0,1] : f(x) = 1\}$ has measure zero. Prove directly (without using any results like monotone or dominated convergence theorem) that

$$\lim_{n\to\infty} \int_0^1 f(x)^n \, dx = 0 .$$

Problem 3. (a) Prove that if $a > 1$ and $k \geq 1$, then $\sum_{n=2}^{\infty} \frac{(\log n)^k}{n^a} < \infty$.

(b) Prove that the function $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$, $x > 1$ is infinitely differentiable in $(1, \infty)$.

Hint: *Use part (a) to prove part (b). You can use part (a) to prove (b), even if you do not know how to prove (a).*

Problem 4. Prove directly (without using the Arzela-Ascoli theorem) that if a sequence of functions $f_n : [0,1] \to \mathbb{R}$ is equi-continuous and convergent at every point $x \in [0,1]$, then f_n is uniformly convergent on $[0,1]$.

Problem 5. Suppose that $f \in C^2(\mathbb{R}^3)$ is constant in a neighborhood of the boundary of a ball $B \subset \mathbb{R}^3$. Prove that

$$\iiint_B (f_{xx} + f_{yy} + f_{zz}) \, dV = 0 .$$

Problem 6. Let $A = [a_{ij}]$ be a symmetric $n \times n$ matrix, i.e. $a_{ij} = a_{ji}$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, i.e. $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

(a) Prove that $\nabla f(x) = 2Ax$.

(b) Prove that if $f|_{S^{n-1}}$ attains maximum at $x_0 \in S^{n-1}$, i.e. $f(x_0) = \sup_{x \in S^{n-1}} f(x)$, then x_0 is an eigenvector of the matrix A, i.e. $Ax_0 = \lambda x_0$ for some $\lambda \in \mathbb{R}$.

Hint: *Use (a) to prove (b). You can use (a) to prove (b) even if you do not know how to prove (a).*