Preliminary Exam in Advanced Calculus

April 2009

Write solutions to different problems on separate sheets. Use scrap paper first to solve problems and write final solutions in a legible form taking care of clarity of presentation.

Problem 1. Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a function $f : \Omega \to \mathbb{R}$ is *locally bounded* if for every $x \in \Omega$ there is a ball $B(x, \varepsilon) \subset \Omega$ such that f is bounded in $B(x, \varepsilon)$. Note that we do not assume continuity of f at any point.

(a) Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is locally bounded, then it is bounded in *every* open ball in $\mathbb{R}^n$.

(b) Give an example of a locally bounded function defined in the unit ball of $\mathbb{R}^n$, $f : B^n(0,1) \to \mathbb{R}$ which is not bounded.

Problem 2. Assume that $f : [0,1] \to [0,1]$ is a continuous function such that the set $\{x \in [0,1] : f(x) = 1\}$ has measure zero. Prove directly (without using any results like monotone or dominated convergence theorem) that

$$\lim_{n\to\infty} \int_0^1 f(x)^n \, dx = 0 .$$

Problem 3. (a) Prove that if $a > 1$ and $k \geq 1$, then $\displaystyle\sum_{n=2}^{\infty} \frac{(\log n)^k}{n^a} < \infty$.

(b) Prove that the function $\zeta(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^x}$, $x > 1$ is infinitely differentiable in $(1, \infty)$.

Hint: *Use part (a) to prove part (b). You can use part (a) to prove (b), even if you do not know how to prove (a).*

Problem 4. Prove directly (without using the Arzela-Ascoli theorem) that if a sequence of functions $f_n : [0,1] \to \mathbb{R}$ is equi-continuous and convergent at every point $x \in [0,1]$, then f_n is uniformly convergent on $[0,1]$.

Problem 5. Suppose that $f \in C^2(\mathbb{R}^3)$ is constant in a neighborhood of the boundary of a ball $B \subset \mathbb{R}^3$. Prove that

$$\iiint_B (f_{xx} + f_{yy} + f_{zz}) \, dV = 0 .$$

Problem 6. Let $A = [a_{ij}]$ be a symmetric $n \times n$ matrix, i.e. $a_{ij} = a_{ji}$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \displaystyle\sum_{i,j=1}^{n} a_{ij} x_i x_j$. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, i.e. $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

(a) Prove that $\nabla f(x) = 2Ax$.

(b) Prove that if $f|_{S^{n-1}}$ attains maximum at $x_0 \in S^{n-1}$, i.e. $f(x_0) = \displaystyle\sup_{x \in S^{n-1}} f(x)$, then x_0 is an eigenvector of the matrix A, i.e. $Ax_0 = \lambda x_0$ for some $\lambda \in \mathbb{R}$.

Hint: *Use (a) to prove (b). You can use (a) to prove (b) even if you do not know how to prove (a).*