Ph.D. PRELIMINARY EXAMINATION

LINEAR ALGEBRA

April 25, 2008

1. Work each problem on separate sheet of paper. Justify all steps in terms of the major results in linear algebra.

2. Label each page with your code number, but not your name, problem number and page number.

CODE NUMBER: ________________

GRADE QUESTIONS: 1. ________ 2. ________

3. ________ 4. ________

5. ________ 6. ________

Linear Algebra 2008

1. Let V be an N-dimensional vector space over $\mathbb{R}$ with basis $\{x_1, x_2, \ldots, x_N\}$. Define

$$y_J = \frac{1}{J}\sum_{j=1}^{J} x_j, \qquad J = 1, \ldots, N.$$

Show that for each J

$$span\{x_1, \ldots, x_J\} = span\{y_1, \ldots, y_J\}.$$

2. Let V be a finite dimensional vector space over $\mathbb{R}$ and $A : V \to V$ a linear transformation.

   (a) If $Range(A) \cap Nullspace(A) = \{0\}$ show that $(Range(A), Nullspace(A))$ reduces A (i.e., for $V = Range(A) \oplus Nullspace(A)$, $A : Range(A) \subset Range(A)$ and $A(Nullspace(A)) \subset Nullspace(A)$).

   (b) Give a 2×2 matrix where

$$Range(A) \cap Nullspace(A) \quad \text{is nontrivial.}$$

3. For $a \neq b \in \mathbb{R}$, consider the matrix

$$A = \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix}.$$

   (a) Diagonalize A.

   (b) Show that

$$A^n = \begin{pmatrix} a^n & b^n - a^n \\ 0 & b^n \end{pmatrix}.$$

4. Let A be an $n \times n$ real matrix with $\det A \neq 0$. Show that A^{-1} can be written as a polynomial in A with real coefficients.

5. Let A be an $n \times n$ real symmetric matrix and $d > 0$.

   (a) Show $\lambda_{min}(A + dI) > \lambda_{min}(A)$

   (b) If D is a diagonal matrix with $d_{ii} > 0$ show

$$\lambda_{min}(A + D) > \lambda_{min}(A).$$

6. Prove that if A is convergent ($\lim_{n\to\infty} A^n = 0$), then the spectral radius of A, $\rho(A)$ satisfies

$$\rho(A) < 1.$$