<u>**Ph.D. Preliminary Examination**</u> (Analysis)

April, 2008

1. Suppose f is a real-valued increasing function on $[0, 1]$. Prove that the set of points in $[0, 1]$ where f is NOT continuous has measure zero.

2. Suppose that a sequence of function $f_1(x), f_2(x), ...$ converges uniformly on $[0, 1]$ to some function $f(x)$. Suppose further that there exists some constant M such that $|f_i(x)| < M$ for all i and x. Prove that the sequence of squares, $f_1^2(x), f_2^2(x), ...$ also converges uniformly, to the function $f^2(x)$.

3. Prove that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a twice differentiable function and $\alpha, \beta > 0$. Assume that for all $x \in \mathbf{R}$, $|f(x)| \leq \alpha$ and $|f''(x)| \leq \beta$. Prove that

$$|f'(x)| \leq 2\sqrt{\alpha\beta}$$

for all $x \in \mathbf{R}$.

5. Let Ω denote the space of all infinite sequences in $\mathbf{R}$. Define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)}$$

where $x = \{x_n\}$ and $y = \{y_n\}$ are in Ω. Prove that (Ω, d) is a metric space.

6. Let f be a continuous function on the unit square $Q = [0, 1] \times [0, 1]$, and for $s \in [0, 1]$ let $g(s) = \max\{f(s, t) : t \in [0, 1]\}$. Show that g is a continuous function on $[0, 1]$.