Preliminary Exam in Linear Algebra

April 2007

Final Exam in Math 2371: Problems 4-6
Preliminary Exam: All problems

Work each problem on separate sheets of paper. Label each page with your name, problem number, and page number. Justify all steps in terms of the major results in linear algebra.

Problem 1 Show that if a linear map *S* on a 2-dimensional linear space *X* is a projection (i.e., $S^2 = S$), then $S = 0$, or $S = I$, or there is a basis *B* such that the matrix representation of *S* with respect to *B* is $S_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Find the basis *B*.

Problem 2: Let *N* be a 3×3 nonzero complex matrix such that $N^2 = 0$. Show that *N* is similar to

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 3: Show that two vectors in a complex Euclidean space are orthogonal if and only if $\|ax + y\|^2 = \|ax\|^2 + \|y\|^2$ for all complex numbers *a*.

Problem 4: Suppose *A* is complex selfadjoint (Hermitian) positive definite $n \times n$ matrix with entries a_{ij}. Suppose that $a_{ii} = 1$ for $i = 1,2,...,n$. Show that $|a_{ij}| < 1$ whenever $i \neq j$.

Problem 5: Let *A* be a complex anti-selfadjoint (skew) matrix and let $B = e^A$. Show that

(a) $\det B = e^{trace(A)}$

(b) $B^* = e^{-A}$

(c) B is unitary.

Problem 6: Let $|.|$ be a norm on $\mathbb{C}^n$. Show that the following function on $n \times n$ complex matrices, $|||A|||_D = \max_{x \neq 0} \frac{|Ax|}{|Dx|}$, is a matrix norm (i.e., submultiplicative norm) for any invertible matrix *D* such that $|||D|||_I \leq 1$.