Preliminary Exam in Advanced Calculus

April 28, 2007; 02:00 – 05:00 pm

Write solutions to different problems on separate sheets. Use scrap paper first to solve problems and write final solutions in a legible form taking care of clarity of presentation.

Problem 1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} a_j = L\,.$$

Problem 2. Determine convergence and/or uniform convergence of the following sequence $f_n(x) = nx(1-x)^n$, $x \in [0,1]$.

Problem 3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function such that $f(U)$ is open whenever $U \subset \mathbb{R}^n$ is open and $f^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^m$ is compact. Prove that $f(\mathbb{R}^n) = \mathbb{R}^m$. (*Hint:* Prove that $f(\mathbb{R}^n)$ is open and closed.)

Problem 4.
(a) State carefully the *definition* of what it means for a function $g : \mathbb{R}^2 \to \mathbb{R}$ to be *differentiable* at $a = (a_1, a_2) \in \mathbb{R}^2$.
(b) Write down an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the directional derivative $f_u(0,0)$ exists in $\mathbb{R}$ for all unit vectors $u \in \mathbb{R}^2$, and yet f is *not* differentiable at $(0,0)$. Also, prove these two facts for your example.
(c) Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x,y) = x^{2/3}y^{2/3}, \text{ for all } (x,y) \in \mathbb{R}^2\,.$$

Prove that g is *differentiable* at $(0,0)$.

Problem 5. Prove that

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy = \sum_{n=1}^{\infty} \frac{1}{n^2}\,.$$

The integral is an improper integral and is understood as $\lim_{t\to 1^-} \int_0^t \int_0^t \ldots$.

Problem 6. Use Green's theorem to prove the following theorem: If the vertices of a polygon, in counterclockwise order, are (x_1,y_1), (x_2,y_2), $\ldots$,(x_n,y_n), then the area of the polygon is

$$A = \frac{1}{2}\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i),$$

where we use notation $x_{n+1} = x_1$, $y_{n+1} = y_1$.