Preliminary Exam in Linear Algebra

April 2006

Part 1: Final Exam in Math 2371 Linear Algebra 2

Work each problem on a separate set of paper. Label the solutions carefully. Fully justify all steps in terms of the major results in linear algebra.

1. Let $||\cdot||$ be a norm on $\mathcal{R}^N$ and let P be an $N \times N$ invertible, real matrix. Define the new norm

$$||x||_P := ||P^{-1}x||.$$

(a) Show that the induced matrix norm is

$$||A||_P := ||P^{-1}AP||.$$

(b) Show that if A is a real $N \times N$ matrix that is diagonalizable, then there exists a norm $||\cdot||_*$ under which (where $spr(A)$ denotes spectral radius of A)

$$||A||_* = spr(A).$$

2. Let D be a real $N \times N$ diagonal matrix, $D = diag(d_1, d_2, \dots, d_N)$ and let P be a real $N \times N$ matrix with $|P_{ij}| \leq 1$ for all i and j. Let

$$A = D + \varepsilon P, \quad 0 \leq \varepsilon \leq 1.$$

Show that for every eigenvalue $\lambda(A)$ there exists d_i such that

$$|\lambda(A) - d_i| \leq N\varepsilon.$$

3. Let V be a finite dimensional, complex inner product space and let $A : V \to V$ be a Hermitean operator. Show that there exist Hermitean operators P and N such that

P is positive semi-definite, N is negative semi-definite,

$$Rge(P) \perp Rge(N), \text{ and } A = P + N.$$

4. Let $V := \mathcal{P}_3$, the polynomials with complex coefficients of degree ≤ 3. Consider the linear operator $A : V \to V$ defined by

$$A : p(t) \mapsto p'(t).$$

(a) Find the matrix representation of A with respect to the basis $\{1, t, t^2, t^3\}$.
(b) Find the Jordan canonical form of this matrix.

Part 2: for students taking the Prelim.

5. Let W_1 and W_2 be subspaces of a vector space V.

(a) Show that there exists a basis of V such that it contains as subsets bases for each of W_1 and W_2.

(b) Give an example of a vector space V and three subspaces W_1, W_2, and W_3 such that any basis of V does not contain as subsets bases for each of the three subspaces.

6. Let A be an $N \times N$ real matrix. If $rank(A) = p$, what is $rank(A^t A)$? Prove your assertion.