Preliminary Analysis Exam

April 2006

You have 180 minutes to complete this exam. There are six problems, 20 points each. Notes, books and electronic devices are not allowed. Be sure to justify your reasoning.

1. Let (M, d) be a metric space.
(a) Define carefully what it means to say (M, d) is (i) sequentially compact, and (ii) complete.

(b) From your definitions, proving all steps, show that if (M, d) is compact then it is complete.

(c) Prove or disprove the following statement: "If $f : (M_1, d_1) \to (M_2, d_2)$ is continuous, and (M_1, d_1) is complete, then (M_2, d_2) is complete".

2.
(a) Give the definition of uniform convergence of a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ on a given interval $I \subseteq \mathbb{R}$.

(b) Test the functions

$$f_n(x) = \frac{nx^2}{1 + n^2x^4}$$

for uniform convergence on: (i) $[0, 1]$ and (ii) $[1, \infty)$.

(c) Compute

$$\lim_{n\to\infty} \int_1^2 \frac{nx^2 dx}{1 + n^2x^4}.$$

3. Let (X, d) be a compact metric space and suppose $f : X \to X$ is an isometry map, i.e. $d(f(x), f(y)) = d(x, y)$ for any x, $y \in X$. Show that f is a bijection. *Hint:* to show that f is onto X, assume that $y_1 \in X \setminus f(X)$ and let $y_2 = f(y_1)$, $y_3 = f(y_2)$ etc.

4. Show that the equations

$$x^2 - y^2 - u^3 + v^3 + 4 = 0$$
$$2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$$

determine functions $u(x,y)$ and $v(x,y)$ near $x = 2$, $y = -1$ that satisfy $u(2,-1) = 2$ and $v(2,-1) = 1$. Compute $\frac{\partial u}{\partial x}(2,-1)$.

5. Let $S = \{(x,y) \in \mathbb{R}^2 : x \text{ is rational}, \ 0 < x < 1, \text{ and if } x = p/m \text{ in lowest form, then } y = k/m, \ k = 1, \ldots, m-1\}$. Let χ_S be the characteristic function of S (i.e. $\chi_S(x,y) = 1$ if $(x,y) \in S$ and $\chi_S(x,y) = 0$ if $(x,y) \notin S$).

Show that

$$\int_0^1 \int_0^1 \chi_S \, dy \, dx = 0, \qquad \text{but} \qquad \int\int_{[0,1]\times[0,1]} \chi_S \quad \text{does not exist.}$$

6. Compute

$$\int_\Gamma (y-z)dx + (z-x)dy + (x-y)dz,$$

where Γ is the ellipse formed by the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1$, oriented counterclockwise if viewed from far out along the positive z-axis.