Linear Algebra Preliminary Exam - April, 2005

Do all four problems

1. (a) Let A and B be real orthogonal matrices. Suppose $\det A = -\det B$. Show that $A+B$ is singular. (Use and justify the relation $A+B = A(B^T + A^T)B$.).

   (b) An $n \times n$ Hadamard matrix A has elements that are all ± 1 and satisfies $A^T A = nI$. Show that $|\det A| = n^{n/2}$.

   (c) The numbers 20604, 53227, 25755, 20927, and 78421 are divisible by 17. Show that the determinant of the following matrix is also divisible by 17:

$$A = \begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{bmatrix}$$

2. Let V be a finite dimensional vector space over the field F and let $T \in L(V)$. Given a subspace W of V, set $T^{-1}(W) = \{x \in V : Tx \in W\}$.

   (a) Show that $T^{-1}(W)$ is a subspace of V.

   (b) Show that $\dim T^{-1}(W) \leq \dim \ker T + \dim W$.

   (c) If $S \in L(V)$, show that $rank ST \geq rank T + rank S - \dim V$ (prove that $\ker ST = T^{-1}(\ker S)$ and use (b)).

3. Let $N \in \mathbb{C}^{n\times n}$ be a nilpotent matrix (i.e., $N^k = 0$ for some $k \in \mathbb{N}$). Show that:

   (a) $\lambda = 0$ is the only eigenvalue of N.

   (b) If $c \in \mathbb{C}$ and $A = cI + N$, then c is the only eigenvalue of A (use (a)).

   (c) If $c \in \mathbb{C}$ and $A = cI + N$ is diagonalizable, then $N = 0$ (use (b)).

   (d) If $c \in \mathbb{C}$, $A = cI + N$ and A^2 is diagonalizable, then $2cN + N^2 = 0$ (prove that $2cN + N^2$ is nilpotent and use (c)).

   (e) If $c \in \mathbb{C}, c \neq 0$ and $2cN + N^2 = 0$, then $N = 0$ (use (b) to show that $2cI + N$ is invertible).

   (f) If $c \in \mathbb{C}, c \neq 0, A = cI + N$ and A^2 is diagonalizable, then A is diagonalizable (use (d) and (e)).

   (g) If $A \in \mathbb{C}^{n\times n}$ is invertible and A^2 is diagonalizable, then A is diagonalizable (use (f) and the Jordan form of A).

   (h) If $A \in \mathbb{C}^{n\times n}$ is not invertible and A^2 is diagonalizable, then A need not be diagonalizable (find a 2×2 example).

4. Let P_3 be the vector space of polynomials $p(x)$ of degree ≤ 3. Define the operator $L : P_3 \to P_3$ as

$$(Lp)(x) = p(0) + xp''(x) + \int_{-1}^{1} (x + y)p(y)dy$$

   (a) Show that L is a linear operator.

   (b) Find a basis for $\ker(L)$ and a basis for $\mathrm{range}(L)$.

   (c) Find the characteristic and minimal polynomials of L.