PRELIMINARY EXAM IN ANALYSIS

April 29, 2005

YOUR NAME (PLEASE PRINT): ____________________

NOTE: If you are taking the final, you need to do all of the problems in PART I only. If you are taking the analysis preliminary, you need to do all of the problems in both PART I and PART II.

PART I: THE FINAL

1. (10 pts) For a function $f : \mathbb{R}^n \to \mathbb{R}^n$, there exist a constant $M > 0$ and a constant $\alpha > 1$ such that, $\|f(y) - f(x)\| \leq M\|y - x\|^\alpha$ for all $x, y \in \mathbb{R}^n$, show that f is differentiable on $\mathbb{R}^n$, $Df(x) = 0$ for every $x \in \mathbb{R}^n$, and f is constant on $\mathbb{R}^n$.

2. (10 pts) Suppose that a C^2 function $f : \mathbb{R}^n \to \mathbb{R}$ has a local minimum at $x = 0$ and $f(0) = 0$. Prove that, for any $x \in \mathbb{R}^n$,
$$f(x) = \int_0^1 (1 - t)\, x^\top D^2 f(tx)\, x\, dt,$$
where $x \in \mathbb{R}^n$ is a column vector and $x^\top$ is the transpose of x.

3. (10 pts) If a function $f : [0, \infty) \to \mathbb{R}$ is nonnegative, integrable, and uniformly continuous, prove that $\lim_{x\to\infty} f(x) = 0$.

4. (10 pts) Suppose that a smooth function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the following property:
$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \quad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}. \tag{1}$$
Prove that, if the Jacobian matrix $Df(x, y) \neq 0$, then f is locally invertible and the inverse function also satisfies property (1).

5. (10 pts) Evaluate the integral $\iint_{\mathbb{R}^2} e^{-x^2-y^2} dxdy$, and show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

6. (10 pts) If $\Omega \subset \mathbb{R}^3$ is a bounded region with smooth boundary $\partial\Omega$, and $\mathbf{n}$ is the unit outward normal vector, then, for any two C^2 functions u, $v : \Omega \to \mathbb{R}$, prove the following identity:
$$\iiint_\Omega (u\Delta v - v\Delta u) dV = \iint_{\partial\Omega} (u\nabla v - v\nabla u) \cdot \mathbf{n}\, dS,$$
where $\Delta u(x, y, z) = u_{xx} + u_{yy} + u_{zz}$.

PART II: THE EXTRA PROBLEMS FOR THE PRELIMINARY

7. (10 pts) Suppose that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and its derivative f' have no common zeros. Prove that f has only finitely many zeros in $[0, 1]$.

8. (10 pts) Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, $f(0) = 0$, and $\lim_{x \to \infty} f(x) = 0$. Prove that, there exists a point $c \in (0, \infty)$ such that $f'(c) = 0$.

9. (10 pts) Prove that the function defined by $f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} \right)^2$ is continuous on $\mathbb{R}$.

10. (10 pts) Prove that, if $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^1 f(x) x^n dx = 0$ for each integer $n \geq 0$, then $f \equiv 0$ on $[0, 1]$.