6. (10 points) Choose any one of the following two problems:

(a) Let \( V \) be a finite dimensional non-trivial vector space over \( \mathbb{C} \) and \( A : V \to V \) be a linear operator. Show that there exists at least one eigenpair of \( A \), i.e., there exist \( \mu \in \mathbb{C} \) and \( x \in V \setminus \{0\} \) such that \( Ax = \mu x \).

(b) Let \( V \) be a finite dimensional vector space over \( \mathbb{C} \) and \( A : V \to V \) be a linear operator satisfying

\[
\ker(\lambda I - A) \cap \text{R}(\lambda I - A) = \{0\}, \quad \forall \lambda \in \mathbb{C},
\]

where \( \ker \) stands for kernel and \( \text{R} \) for range. Show that there is a basis \( \{v_1, \cdots, v_n\} \) of \( V \) such that each \( v_j, j = 1, \cdots, n \), is an eigenvector of \( A \). (You can use the result from the previous problem (a).)

7. (2 \times 20 pts) Choose any two of the following three problems.

(a) Let \( p(x) \) be the characteristic polynomial of \( A \in M_n(\mathbb{R}) \) and \( p(x) = p_1(x)p_2(x) \) where \( p_1 \) and \( p_2 \) are relatively prime polynomials with real coefficients. Show that

\[
\mathbb{R}^n = \ker(p_1(A)) \oplus \ker(p_2(A))
\]

where \( \ker \) stands for the kernel.

(b) Let \( \langle f, g \rangle = \int_0^1 f(t)g(t)dt \) for any continuous functions \( f, g \) defined on \([0, 1]\). Let \( E_k \) be the Euclidean space of all polynomials of real variable \( t \), with real coefficients, of degree \( \leq k \), and equipped with the inner product \( \langle \cdot, \cdot \rangle \). Let \( P_n : E_{n+1} \to E_n \) be the orthogonal projection from \( E_{n+1} \) to \( E_n \) and \( A : E_n \to E_{n+1} \) be defined by \( Ap(t) = tp(t) \) for all \( p \in E_n \). Show the following:

(i) \( P_nA : E_n \to E_n \) is a self-adjoint operator on \( E_n \).

(ii) \( AP_n : E_{n+1} \to E_{n+1} \) is not a self-adjoint operator on \( E_{n+1} \).

(c) Let \( \mathbb{F} \) be a field and \( A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in M_4(\mathbb{F}) \). Prove the following:

(i) there exists \( \xi \in \mathbb{F}^4 \) such that \( \{\xi, A\xi, A^2\xi, A^3\xi\} \) is a basis of \( \mathbb{F}^4 \);

(ii) \( A^4\xi = \xi - 2A^2\xi \) and \( A^4 = I - 2A^2 \);

(iii) \( B \in M_4(\mathbb{F}) \) is a polynomial of \( A \) if and only if \( AB = BA \).