6. (10 points) <u>*Choose any one of the following two problems:*</u>

(a) Let V be a finite dimensional non-trivial vector space over $\mathbb{C}$ and $\mathbf{A} : V \to V$ be a linear operator. Show that there exists at least one eigenpair of $\mathbf{A}$, i.e., there exist $\mu \in \mathbb{C}$ and $\mathbf{x} \in V \setminus \{\mathbf{0}\}$ such that $\mathbf{A}\mathbf{x} = \mu\mathbf{x}$.

(b) Let V be a finite dimensional vector space over $\mathbb{C}$ and $\mathbf{A} : V \to V$ be a linear operator satisfying

$$\ker(\lambda\mathbf{I} - \mathbf{A}) \cap \mathrm{R}(\lambda\mathbf{I} - \mathbf{A}) = \{\mathbf{0}\} \qquad \forall\lambda \in \mathbb{C},$$

where ker stands for kernel and R for range. Show that there is a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ of V such that each $\mathbf{v}_j, j = 1, \cdots, n$, is an eigenvector of $\mathbf{A}$. (You can use the result from the previous problem (a).)

7. (2×20 pts) <u>*Choose any two of the following three problems.*</u>

(a) Let $p(x)$ be the characteristic polynomial of $A \in M_n(\mathbb{R})$ and $p(x) = p_1(x)p_2(x)$ where p_1 and p_2 are relatively prime polynomials with real coefficients. Show that

$$\mathbb{R}^n = \ker(p_1(A)) \oplus \ker(p_2(A))$$

where ker stands for the kernel.

(b) Let $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$ for any continuous functions f, g defined on $[0, 1]$. Let E_k be the Euclidean space of all polynomials of real variable t, with real coefficients, of degree $\leq k$, and equipped with the inner product $\langle\cdot,\cdot\rangle$. Let $\mathbf{P}_n : E_{n+1} \to E_n$ be the orthogonal projection from E_{n+1} to E_n and $\mathbf{A} : E_n \to E_{n+1}$ be defined by $\mathbf{A}p(t) = tp(t)$ for all $p \in E_n$. Show the following:

(i) $\mathbf{P}_n\mathbf{A} : E_n \to E_n$ is a self-adjoint operator on E_n.

(ii) $\mathbf{A}\mathbf{P}_n : E_{n+1} \to E_{n+1}$ is not a self-adjoint operator on E_{n+1}.

(c) Let $\mathbb{F}$ be a field and $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in M_4(\mathbb{F})$. Prove the following:

(i) there exists $\xi \in \mathbb{F}^4$ such that $\{\xi, A\xi, A^2\xi, A^3\xi\}$ is a basis of $\mathbb{F}^4$;

(ii) $A^4\xi = \xi - 2A^2\xi$ and $A^4 = I - 2A^2$;

(iii) $B \in M_4(\mathbb{F})$ is a polynomial of A if and only if $AB = BA$.