7. (20 points) Determine for which $p \in \mathbb{R}$ the sequence of functions

\[ f_n(x) = \frac{x e^{-x^2/n^2}}{n^p} \]

converges uniformly on $\mathbb{R}$, as $n \to \infty$.

8. (20 points) In $\mathbb{R}^2$, define the sets

\[ T_1 = \{(x, y) : (x - 2)^2 + y^2 = 4, y \leq 0\}, \text{ and} \]
\[ T_2 = \{(x, y) : y = f(x)\}, \text{ where } f(0) = f(4) = 0, f(x) \geq 0, \text{ and } \int_0^4 f(x) \, dx = 1/2. \]

a) Find a parametrization of $T_1$, oriented in the counterclockwise direction.

b) Let $D \subset \mathbb{R}^2$ be the region bounded below by $T_1$ and above by $T_2$. Compute the oriented line integral of $F = (x^3, x - y^2)$ around $\partial D$, oriented in the counterclockwise direction.

c) Show that for $g$ differentiable on $[0, L]$ with $g'(x) > 0$, the arc length of the explicit curve defined by $\{y = g(x) : 0 \leq x \leq L\}$ is less than or equal to $L + g(L) - g(0)$.

9. (20 points) For all of this problem, let $A \subset \mathbb{R}^n$ be open and bounded with volume and let $g : A \to \mathbb{R}^n$ be a $C^1$ and one-to-one function such that $g(A)$ has volume. The change of variables formula is the relation

\[ \int_{g(A)} f = \int_A (f \circ g) |Jg|, \quad (1) \]

which holds for $f$ any integrable, bounded function from $g(A)$ to $\mathbb{R}$ if $B = \{x \in A : Jg(x) = 0\}$ is empty. The objective of this problem is to use your knowledge of Riemann integration and volume, together with the fact that (1) holds for $B$ empty, to extend this formula to the case where $B$ is nonempty.

a) Suppose first that $C \subset A$ such that $Jg(x) \neq 0$ on $C$, which implies that equation (1) holds for any bounded, integrable function $f : g(C) \to \mathbb{R}$. If $S$ is a set of measure zero in $g(C)$, then the inverse image of $S$ with respect to $g$ also has measure zero, which you may use in part b). Prove this fact in the special case when there exist constants $m_1, m_2$ such that

\[ 0 < m_1 \leq |Jg(x)| \leq m_2 \quad (2) \]
on $C$.

b) Assume that $\{x \in A : Jg(x) \neq 0\}$ has volume. Extend the change of variables formula (1) to the case when $B$ is nonempty. You may use the fact (not shown here) that $g(B)$ has measure zero in your proof. You may also use the fact that (1) holds on subsets of $A$ on which $Jg(x) \neq 0$. 