7. (20 points) Determine for which $p \in \mathbb{R}$ the sequence of functions

$$f_n(x) = \frac{xe^{-x^2/n^2}}{n^p}$$

converges uniformly on $\mathbb{R}$, as $n \to \infty$.

8. (20 points) In $\mathbb{R}^2$, define the sets

$$T_1 = \{(x,y) : (x-2)^2 + y^2 = 4, y \le 0\}, \text{ and}$$
$$T_2 = \{(x,y) : y = f(x)\}, \text{ where } f(0) = f(4) = 0,\ f(x) \ge 0, \text{ and } \textstyle\int_0^4 f(x)\,dx = 1/2.$$

a) Find a parametrization of T_1, oriented in the counterclockwise direction.

b) Let $D \subset \mathbb{R}^2$ be the region bounded below by T_1 and above by T_2. Compute the oriented line integral of $F = (x^3, x - y^2)$ around ∂D, oriented in the counterclockwise direction.

c) Show that for g differentiable on $[0, L]$ with $g'(x) > 0$, the arc length of the explicit curve defined by $\{y = g(x) : 0 \le x \le L\}$ is less than or equal to $L + g(L) - g(0)$.

9. (20 points) For all of this problem, let $A \subset \mathbb{R}^n$ be open and bounded with volume and let $g : A \to \mathbb{R}^n$ be a $\mathcal{C}^1$ and one-to-one function such that $g(A)$ has volume. The change of variables formula is the relation

$$\int_{g(A)} f = \int_A (f \circ g)|Jg|, \tag{1}$$

which holds for f any integrable, bounded function from $g(A)$ to $\mathbb{R}$ if $B = \{x \in A : Jg(x) = 0\}$ is empty. The objective of this problem is to use your knowledge of Riemann integration and volume, together with the fact that (1) holds for B empty, to extend this formula to the case where B is nonempty.

a) Suppose first that $C \subset A$ such that $Jg(x) \neq 0$ on C, which implies that equation (1) holds for any bounded, integrable function $f : g(C) \to \mathbb{R}$. If S is a set of measure zero in $g(C)$, then the inverse image of S with respect to g also has measure zero, which you may use in part b). Prove this fact in the special case when there exist constants m_1, m_2 such that

$$0 < m_1 \le |Jg(x)| \le m_2 \tag{2}$$

on C.

b) Assume that $\{x \in A : Jg(x) \neq 0\}$ has volume. Extend the change of variables formula (1) to the case when B is nonempty. You may use the fact (not shown here) that $g(B)$ has measure zero in your proof. You may also use the fact that (1) holds on subsets of A on which $Jg(x) \neq 0$.