LINEAR ALGEBRA FINAL AND PRELIMINARY EXAMS   Apr 23, 2004

The Math 2371 Final Examination consists of Exercises 1-6.

Students taking the preliminary examination should also work Exercise 7.

In the following, n is a generic positive integer, $\mathbb{F}$ a generic field, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers, $\mathbf{i} = \sqrt{-1}$, $M_n(R)$ the set of all $n \times n$ matrices with entries in a ring R, I the identity matrix of appropriate rank, and $\mathbf{I}$ the identity map.

1. (10pts) Let $A = \begin{pmatrix} x-1 & 1 \\ 0 & x-1 \end{pmatrix}$. Find invertible 2×2 matrices P, Q (whose entries are polynomials of x) such that PAQ is a diagonal matrix.

2. (10pts) Prove that for any $A, B \in M_n(\mathbb{F})$, AB and BA have the same set of eigenvalues.

3. (10pts) Which of the following matrices is *unitarily* similar to a diagonal matrix? (You have to provide a reason such as "because the matrix is Hermitian" for your conclusion.)

$$\begin{pmatrix} 1 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

4. (10pts) Let $A \in M_n(\mathbb{R})$ be symmetric positive definite, i.e. $A = A^T$ and $x^T A x > 0$ for all non-trivial $x \in \mathbb{R}^n$. Define $\langle x, y \rangle = x^T A y$ for all $x, y \in \mathbb{R}^n$. Directly prove the Cauchy–Schwarz inequality:

$$\langle x, y \rangle \le \sqrt{\langle x, x \rangle}\sqrt{\langle y, y \rangle} \qquad \forall\, x, y \in \mathbb{R}^n.$$

5. (10pts) <u>*Choose any one of the following two problems.*</u>

   (a) Let E_3, a vector space over $\mathbb{R}$, be the set of all polynomials of real coefficients and degree ≤ 3. Define $\mathbf{T} : E_3 \to E_3$ by

   $$\mathbf{T}p = p'' + 2p' + p \qquad \forall\, p \in E_3,$$

   where $'$ denotes differentiation. Find all the eigenvalues, including their multiplicity, of $\mathbf{T}$.

   (b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ and define $\mathbf{T} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

   $$\mathbf{T}B = AB - BA \qquad \forall\, B \in M_2(\mathbb{C}).$$

   Find all the eigenvalues, including their multiplicity, of the linear operator $\mathbf{T}$.