PRELIMINARY EXAM IN ANALYSIS 4/21/2004

Name:____

You have 180 minutes to complete this exam. Be sure to justify your reasoning. In all problems with multiple parts, you may use earlier parts in the answers of later parts, even if you cannot manage to complete the earlier parts. The first two questions are worth 10 points each. The next seven questions are worth 20 points each. Do as much of the exam as you can.

1. (10 points) Prove or disprove the claim that the function $f(x,y):\mathbb{R}^2\to\mathbb{R}$ is continuous at $(0,0)$, if

$$f(x,y)=\begin{cases} e^{-xy/(x^2+y^2)}, & (x,y)\neq(0,0),\\ 1, & (x,y)=(0,0).\end{cases}$$

2. (10 points) Let $A\subset\mathbb{R}^n$ be open and let $f:A\to\mathbb{R}^m$, for some natural numbers m,n. Suppose there is a natural number $p>1$ such that $||f(x)||\leq||x||^p$ on A. Prove that f is differentiable at $0\in A$, and determine $Df(0)$.

3. (20 points) Let $Tu(t)=\int_0^1 K(s,t)f(s,u(s))\,ds$ be a function of $u\in\mathcal{C}([0,1],[0,1])$. Assume that $K,f\in\mathcal{C}([0,1]\times[0,1],\mathbb{R})$. Derive a condition for the existence of a solution to the equation $u=Tu+c$, for constant $c\neq0$, under the additional assumption that f is continuously differentiable with respect to u.

4. (20 points) On $\mathcal{C}([0,1],\mathbb{R})$, let $||f||=\sup\{|f(t)r(t)|:t\in[0,1]\}$ for some fixed function $r(t)\in\mathcal{C}([0,1],\mathbb{R})$.

   a) State and verify necessary and sufficient conditions on $r(t)$ such that $||\cdot||$ defines a norm on $\mathcal{C}([0,1],\mathbb{R})$.

   b) For $r(t)=e^{-at}$, where $a>0$ is a constant, is $\mathcal{C}([0,1],\mathbb{R})$ complete under this norm? Prove this or show that it fails.

5. (20 points) Evaluate the integral

$$\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{\cos^2 nx}{n^2+n}\,dx.$$

6. (20 points)

   a) Let A be a closed set in $\mathbb{R}^n$ and let $B\subset A$ be open relative to A. Prove that $A\setminus B$ is closed.

   b) Suppose that f is a one-to-one, continuous function from a compact subset A of $\mathbb{R}^n$ to an arbitrary metric space M. Suppose that for some $B\subset A$, $f(B)$ is open in M. Show that $f(A\setminus B)$ is compact in M.