Ph.D. PRELIMINARY EXAMINATION

PART I – LINEAR ALGEBRA

April 21, 2003

1. Follow the instructions on page 1.
2. Indicate below which questions you wish to have graded.
3. Use of soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.
4. Write your name on this sheet.
   Confine your answers to the rectangular area indicated on the answer sheets.

NAME: ______________________________

Part I

GRADE QUESTIONS: 1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________ 7. ________

8. ________

Part II

GRADE QUESTIONS: 1. ________ 2. ________ 3. ________

4. ________ 5. ________

April 2003
Linear
Algebra

PRELIMINARY/ FINAL EXAM: LINEAR ALGEBRA

This constitutes both the final exam for MATH 2371 and the preliminary examination. If you are only taking the FINAL exam, then just do six out of eight problems from the first part. If you are taking the PRELIM, then do six out of eight problems from the first part and three out of five problems from the second part. All problems are worth 12 points.

PART I. Complete SIX questions for the final exam.

1. Show that a complex $n * n$ matrix T is normal if and only if $T = T_1 + iT_2$ where T_1, T_2 are commuting Hermitian matrices.

2. Let M be an $n * n$ complex matrix.

(a) Show that M^*M is positive semidefinite.

(b) Show that M^*M is positive definite if and only if M is invertible.

3. Let M be an $n * n$ real matrix. Recall that $||A||_2$ (the 2-norm of A) is defined by $max_{\vec{y}\neq 0}\frac{|A\vec{y}|}{|\vec{y}|}$ where $|\vec{y}|$ is the usual Euclidean length. Prove that $||A||_2 = max|\vec{x}^T A\vec{y}| : |\vec{x}| = |\vec{y}| = 1$. Here ^{T} denotes transpose.

4.

(a) Is it true that if A, B are $n * n$ complex matrices, then $e^{A+B} = e^A e^B$? Prove or give a counterexample.

(b) Show that if $A^* = -A$ then e^A is unitary.

5. Show that if A is Hermitian and B is positive definite, then the smallest eigenvalue of $A + B$ is larger than the smallest eigenvalue of A.

6.

(a) Show that if a complex 2 by 2 matrix (a_{ij}) is positive definite, then $a_{11}a_{22} > |a_{12}|^2$.

(b) Show that if an complex $n * n$ matrix is positive definite, then $a_{ii}a_{jj} > |a_{ij}|^2$ for all $i, j = 1, \cdots, n$.

7. Show that if A, B are commuting $n * n$ real matrices, then each eigenvalue of AB is a product of some eigenvalue of A with some eigenvalue of B.

8. Let A be an $n * n$ real symmetric matrix.

(a) Show that the eigenvalues of A are real.

(b) Show that if A is also orthogonal, then the eigenvalues of A must be ± 1.

(c) Let A = $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find an orthogonal matrix Q so that $Q^{-1}AQ$ is diagonal.

PART II. If you are taking the prelim., you must also do three out of five problems from this part.

1.

(a) Prove that for an $n \times n$ complex matrix A in Jordan form, there is a permutation matrix P such that PAP^{-1} is equal to the transpose of A, and satisfying $P = P^{-1}$.

(b) Prove that an $n \times n$ complex matrix A is similar to its transpose.

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2. True-False (A short outline of a proof using well-known results, or a counterexample, is needed for each answer. In the false cases there are 2 by 2 counterexamples).

(a) (2 pts.) If B is formed by interchanging two rows of A, then B is similar to A

(b) (2 pts.) If a triangular matrix is similar to a diagonal matrix, it is already diagonal

(c) (2 pts.) If A and B are diagonalizable, so is AB.

(d) (2 pts.) Every invertible matrix can be diagonalized.

(e) (2 pts.) A square matrix A is never similar to $A - I$.

(f) (2 pts.) If we know that a 2×2 matrix A has all eigenvalues 0 then we can determine the eigenvalues of $A^T A$.

3. Prove that a set of eigenvectors of an $n \times n$ matrix A which correspond to distinct eigenvalues is linearly independent.

4. The set $x_1 = (1, 1, 1), x_2 = (1, 1, -1), x_3 = (1, -1, -1)$ forms a basis of R^3. If y_1, y_2, y_3 is a dual basis with respect to the standard inner product, and if $x = (0, 1, 0)$, find $< x, y_1 >, < x, y_2 >, < x, y_3 >$.

5. Prove that if V_1, V_2 are subspaces of a finite dimensional vector space V, then $dim(V_1 + V_2) = dim(V_1) + dim(V_2) - dim(V_1 \cap V_2)$.