Analysis

April 2003

MATH 1540

Saturday, April 26, 2003

# FINAL

The questions are independent: Not answering a question has no impact on your ability to answer any subsequent question. However, you may have to use the *result* of question # 1,..., i, to answer question # i+1 and you may *or may not* be told which previous question you should use.

Throughout the problem, $||\cdot||$ denotes the euclidian norm on $\mathbb{R}^n$ and the "dot" notation is used for the euclidian inner product on $\mathbb{R}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function such that

$$D^2 f(x)(h, h) > 0, \quad \forall x \in \mathbb{R}^n, \quad \forall h \in \mathbb{R}^n \backslash \{0\}. \tag{1}$$

(i) Show that $f(y) > f(x) + Df(x)(y - x)$ for every $x, y \in \mathbb{R}^n, y \neq x$.

(ii) Deduce from (i) that $Df(x)$ cannot vanish at more than one point $x$.

In (iii) and (iv) below, it is assumed that, in addition to (1),

$$\lim_{||x|| \to \infty} f(x) = \infty. \tag{2}$$

(iii) Show that $f$ is bounded from below and has at least one global minimum. (Consider a sequence $(x_k) \subset \mathbb{R}^n$ such that $f(x_k) \to \inf_{x \in \mathbb{R}^n} f(x)$.)

(iv) Deduce from (ii) and (iii) that the equation $\nabla f(x) = 0$ has one and only one solution.

(v) Given $z \in \mathbb{R}^n$, define $g(x) = f(x) - z \cdot x$. Show that $g$ is $C^2$ and that $D^2 g(x)(h, h) > 0$ for all $x \in \mathbb{R}^n$ and all $h \in \mathbb{R}^n \backslash \{0\}$.

In (vi) and (vii) below, it is assumed that, in addition to (1),

$$\lim_{||x|| \to \infty} \frac{f(x)}{||x||} = \infty. \tag{3}$$

(vi) Show that $\lim_{||x|| \to \infty} g(x) = \infty$.

(vii) Deduce from the above that, given $z \in \mathbb{R}^n$, the equation $\nabla f(x) = z$ has a unique solution $x \in \mathbb{R}^n$.

From now on, $n = 2$ and $f(= f(u, v))$ is the function given by

$$f(u, v) = u^2 + \sin u \sin v + v^2.$$

(viii) Find the Hessian matrix $H_f(u, v)$ of $f$ and show that $\mathrm{Tr} H_f(u, v) > 0$ and $\det H_f(u, v) \neq 0$ for every $(u, v) \in \mathbb{R}^2$, where Tr and det denote the trace and determinant, respectively. Deduce from this that $H_f(u, v)h \cdot h > 0$ for every $(u, v) \in \mathbb{R}^2$ and every $h \in \mathbb{R}^2 \setminus \{0\}$.

(ix) Show that the system

$$\begin{cases} 2u + \cos u \sin v = b, \\ 2v + \sin u \cos v = c, \end{cases}$$

has a unique solution $(u, v) \in \mathbb{R}^2$ for every $(b, c) \in \mathbb{R}^2$.

PRELIM

One hour. Do two problems out of three.

**1)** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable.

(a) Show that if $Df(x)$ is *not* invertible for some $x \in \mathbb{R}^n$, there is a sequence $(x_k) \subset \mathbb{R}^n$ such that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} \frac{f(x_k)-f(x)}{||x_k-x||} = 0$. (Use the definition of differentiability and recall that a linear mapping on $\mathbb{R}^n$ is invertible if and only if it is one to one.)

(b) Let $x_0 \in \mathbb{R}^n$ be given and suppose that there are constants $\delta > 0$ and $M \geq 0$ such that $||f(x) - f(x_0)|| \leq M||x - x_0||$ whenever $||x - x_0|| < \delta$. Show that $||Df(x_0)h|| \leq (M + \varepsilon)||h||$ for every $h \in \mathbb{R}^n$ and every $\varepsilon > 0$ and hence that $||Df(x_0)h|| \leq M||h||$ for every $h \in \mathbb{R}^n$.

Suppose now that there is a constant $\alpha > 0$ such that $||f(y) - f(x)|| \geq \alpha||y - x||$ for every $x, y \in \mathbb{R}^n$.

(c) Deduce from (a) that $Df(x)$ is invertible for every $x \in \mathbb{R}^n$.

(d) Show that, if $f$ is $C^1$, then $||Df(x)^{-1}h|| \leq \frac{1}{\alpha}||h||$ for every $h \in \mathbb{R}^n$ and every $x \in \mathbb{R}^n$. (Hint: By (c), $Df(x)^{-1}$ exists. Then, make *correct* use of (b) and of the inverse function theorem).

**2)** Let $(M, d)$ be a compact metric space and let $C(M)$ denote the set of real-valued continuous functions on $M$ equipped with the distance $\rho(f, g) = \max_{x\in M} |f(x)-g(x)|$. Let $x_* \in M$ be chosen once and for all.

(a) Show that the mapping $f \in C(M) \mapsto f(x_*) \in \mathbb{R}$ is continuous.

(b) Show that $C_*(M) = \{f \in C(M) : f(x_*) = 0\}$ is an algebra and a closed subset of $C(M)$. (Use (a).)

(c) Let $\mathcal{A}$ denote a subalgebra of $C_*(M)$. Show that the set $\mathcal{B} = \{\psi = \varphi + c : \varphi \in \mathcal{A}, c \in \mathbb{R}\}$ is a subalgebra of $C(M)$ and that $\mathcal{A}$ separates the points of $M$ if and only if $\mathcal{B}$ separates the points of $M$.

(d) Deduce from (c) that if $\mathcal{A}$ separates the points of $M$, then $\mathcal{A}$ is dense in $C_*(M)$.

**3)** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be of class $C^1$.

(i) Show that if $Df(x)$ is onto $\mathbb{R}^m$ for every $x \in \mathbb{R}^n$, then $f(\mathbb{R}^n)$ is an open subset of $\mathbb{R}^m$.

(ii) Show that if $f^{-1}(K)$ is a compact subset of $\mathbb{R}^n$ whenever $K \subset \mathbb{R}^m$ is a compact subset, then $f(\mathbb{R}^n)$ is a closed subset of $\mathbb{R}^m$.

(iii) Deduce from (i) and (ii) that if $Df(x)$ is onto $\mathbb{R}^m$ for every $x \in \mathbb{R}^n$ and if $\lim_{||x||\to\infty} ||f(x)|| = \infty$, then $f$ is onto $\mathbb{R}^m$.