The questions are independent: Not answering a question has no impact on your ability to answer any subsequent question. However, you may have to use the result of question \# 1,.., i, to answer question \# i+1 and you may or may not be told which previous question you should use.

Throughout the problem, ||·|| denotes the euclidian norm on \( \mathbb{R}^n \) and the "dot" notation is used for the euclidian inner product on \( \mathbb{R}^n \).

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a \( C^2 \) function such that

\[
D^2 f(x)(h, h) > 0, \quad \forall x \in \mathbb{R}^n, \quad \forall h \in \mathbb{R}^n \setminus \{0\}.
\]  

(1)

(i) Show that \( f(y) > f(x) + Df(x)(y - x) \) for every \( x, y \in \mathbb{R}^n \), \( y \neq x \).

(ii) Deduce from (i) that \( Df(x) \) cannot vanish at more than one point \( x \).

In (iii) and (iv) below, it is assumed that, in addition to (1),

\[
\lim_{||x|| \to \infty} f(x) = \infty.
\]  

(2)

(iii) Show that \( f \) is bounded from below and has at least one global minimum. (Consider a sequence \( (x_k) \subset \mathbb{R}^n \) such that \( f(x_k) \to \inf_{x \in \mathbb{R}^n} f(x) \).

(iv) Deduce from (ii) and (iii) that the equation \( \nabla f(x) = 0 \) has one and only one solution.

(v) Given \( z \in \mathbb{R}^n \), define \( g(x) = f(x) - z \cdot x \). Show that \( g \) is \( C^2 \) and that \( D^2 g(x)(h, h) > 0 \) for all \( x \in \mathbb{R}^n \) and all \( h \in \mathbb{R}^n \setminus \{0\} \).

In (vi) and (vii) below, it is assumed that, in addition to (1),

\[
\lim_{||x|| \to \infty} \frac{f(x)}{||x||} = \infty.
\]  

(3)
(vi) Show that \( \lim_{||x|| \to \infty} g(x) = \infty \).

(vii) Deduce from the above that, given \( z \in \mathbb{R}^n \), the equation \( \nabla f(x) = z \) has a unique solution \( x \in \mathbb{R}^n \).

From now on, \( n = 2 \) and \( f(= f(u,v)) \) is the function given by

\[
f(u,v) = u^2 + \sin u \sin v + v^2.
\]

(viii) Find the Hessian matrix \( H_f(u,v) \) of \( f \) and show that \( \text{Tr} H_f(u,v) > 0 \) and \( \det H_f(u,v) \neq 0 \) for every \( (u,v) \in \mathbb{R}^2 \), where \( \text{Tr} \) and \( \det \) denote the trace and determinant, respectively. Deduce from this that \( H_f(u,v) h \cdot h > 0 \) for every \( (u,v) \in \mathbb{R}^2 \) and every \( h \in \mathbb{R}^2 \setminus \{0\} \).

(ix) Show that the system

\[
\begin{cases}
2u + \cos u \sin v = b, \\
2v + \sin u \cos v = c,
\end{cases}
\]

has a unique solution \( (u,v) \in \mathbb{R}^2 \) for every \( (b,c) \in \mathbb{R}^2 \).
PRELIM

One hour. Do two problems out of three.

1) Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be differentiable.
   (a) Show that if \( Df(x) \) is not invertible for some \( x \in \mathbb{R}^n \), there is a sequence \( (x_k) \subset \mathbb{R}^n \) such that \( \lim_{k \to \infty} x_k = x \) and \( \lim_{k \to \infty} \frac{f(x_k) - f(x)}{||x_k - x||} = 0 \). (Use the definition of differentiability and recall that a linear mapping on \( \mathbb{R}^n \) is invertible if and only if it is one to one.)
   (b) Let \( x_0 \in \mathbb{R}^n \) be given and suppose that there are constants \( \delta > 0 \) and \( M \geq 0 \) such that \( ||f(x) - f(x_0)|| \leq M||x - x_0|| \) whenever \( ||x - x_0|| < \delta \). Show that \( ||Df(x_0)h|| \leq (M + \varepsilon)||h|| \) for every \( h \in \mathbb{R}^n \) and every \( \varepsilon > 0 \) and hence that \( ||Df(x_0)h|| \leq M||h|| \) for every \( h \in \mathbb{R}^n \).
   Suppose now that there is a constant \( \alpha > 0 \) such that \( ||f(y) - f(x)|| \geq \alpha||y - x|| \) for every \( x, y \in \mathbb{R}^n \).
   (c) Deduce from (a) that \( Df(x) \) is invertible for every \( x \in \mathbb{R}^n \).
   (d) Show that, if \( f \) is \( C^1 \), then \( ||Df(x)^{-1}h|| \leq \frac{1}{\alpha}||h|| \) for every \( h \in \mathbb{R}^n \) and every \( x \in \mathbb{R}^n \). (Hint: By (c), \( Df(x)^{-1} \) exists. Then, make correct use of (b) and of the inverse function theorem.)

2) Let \((M, d)\) be a compact metric space and let \( C(M) \) denote the set of real-valued continuous functions on \( M \) equipped with the distance \( \rho(f, g) = \max_{x \in M} |f(x) - g(x)| \).
   Let \( x_* \in M \) be chosen once and for all.
   (a) Show that the mapping \( f \in C(M) \mapsto f(x_*) \in \mathbb{R} \) is continuous.
   (b) Show that \( C_*(M) = \{ f \in C(M) : f(x_*) = 0 \} \) is an algebra and a closed subset of \( C(M) \). (Use (a).)
   (c) Let \( \mathcal{A} \) denote a subalgebra of \( C_*(M) \). Show that the set \( \mathcal{B} = \{ \psi = \varphi + c : \varphi \in \mathcal{A}, c \in \mathbb{R} \} \) is a subalgebra of \( C(M) \) and that \( \mathcal{A} \) separates the points of \( M \) if and only if \( \mathcal{B} \) separates the points of \( M \).
   (d) Deduce from (c) that if \( \mathcal{A} \) separates the points of \( M \), then \( \mathcal{A} \) is dense in \( C_*(M) \).

3) Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be of class \( C^1 \).
   (i) Show that if \( Df(x) \) is onto \( \mathbb{R}^m \) for every \( x \in \mathbb{R}^n \), then \( f(\mathbb{R}^n) \) is an open subset of \( \mathbb{R}^m \).
   (ii) Show that if \( f^{-1}(K) \) is a compact subset of \( \mathbb{R}^n \) whenever \( K \subset \mathbb{R}^m \) is a compact subset, then \( f(\mathbb{R}^n) \) is a closed subset of \( \mathbb{R}^m \).
   (iii) Deduce from (i) and (ii) that if \( Df(x) \) is onto \( \mathbb{R}^m \) for every \( x \in \mathbb{R}^n \) and if \( \lim_{||x|| \to \infty} ||f(x)|| = \infty \), then \( f \) is onto \( \mathbb{R}^m \).