Ph.D. PRELIMINARY EXAMINATION

PART I – LINEAR ALGEBRA

April 26, 2002

1. Follow the instructions on page 1.
2. Indicate below which questions you wish to have graded.
3. Use of soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.
4. Put your code number, but not your name, on each answer sheet that you submit. Confine your answers to the rectangular area indicated on the answer sheets.

CODE NUMBER: ________________

Part I

GRADE QUESTIONS: 1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________ 7. ________

8. ________ 9. ________

Part II

GRADE QUESTIONS: 1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________

PRELIMINARY/FINAL EXAM : LINEAR ALGEBRA

Spring '02

This constitutes both the final exam for **MATH 2371** and the preliminary examination. If you are only taking the **FINAL** exam, then just do six problems from the first part. If you are taking the **PRELIM**, then choose any nine problems from the set of 15. If you are taking both the **PRELIM & FINAL**, then do 6 from part I and 3 from part II.

You have two hours to complete the six of the first nine problems for the FINAL EXAM and one additional hour to complete three of the last five problems for the PRELIM.

PART I. Complete **SIX** questions for the final exam.

1. Suppose that A is a positive definite $n \times n$ complex matrix. Prove the following things:
   (a) The diagonal entries are real and positive.
   (b) Let $(x, y)_A \equiv (x, Ay)$. Prove that $(x, y)_A$ is an inner product on C^n.

2. Let A be an $n \times n$ complex matrix.
   (a) Prove that if $A + A^*$ is positive definite, the the eigenvalues of A have positive real parts.
   (b) Without explicitly computing them, show that the eigenvalues of the matrix below have positive real parts.

$$A = \begin{bmatrix} 2 & -9 & 2 \\ 10 & 1 & -5 \\ 1 & 4 & 3 \end{bmatrix}$$

3. Find e^{At} for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

4. Let $A^* = -A$. Prove that the eigenvalues of A have zero real parts and that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

5. Let $A = (a_{ij})$ be a real square matrix, λ a simple eigenvalue. Let $Av = \lambda v$ and $A^T w = \lambda w$. Recall that the eigenvalues and eigenvectors depend differentiably on the entries of the matrix. Prove that

$$\frac{\partial \lambda}{\partial a_{ij}} = \frac{w_i v_j}{w^T v}$$

6. Suppose that A is positive definite. Show that the maximal eigenvalue of A is given by:

$$\max_{x^* A x = 1} \frac{1}{x^* x}$$

7. Let

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

   (a) Find the eigenvalues and eigenvectors of A
   (b) Find the unique positive definite matrix square root of A.
   (c) Find a matrix B such that $A = \exp(B)$.

8. Suppose $a, b, c, d > 0$ and $s = a + b + c + d$. Consider the matrix:

$$A = \begin{bmatrix} -1-s & a & b & c & 0 & d \\ a & -s & b & c & d & 0 \\ b & c & -s & d & 0 & a \\ c & d & 0 & -s & a & b \\ d & 0 & a & b & -s & c \\ 0 & a & b & c & d & -s-1 \end{bmatrix}$$

   Prove that all eigenvalues of A have negative real parts and thus that A is invertible.

9. A *probability matrix* is matrix whose entries are all positive and whose column sums add up to 1. A *probability vector* is a vector with positive entries and whose entries sum to 1. Prove the following:

   - If A, B are probability matrices, then so is AB.
   - Prove that the spectral radius of probability matrix is less than or equal to 1. (The spectral radius is the maximum of the absolute values of the eigenvalues.)
   - Prove that if for every probability vector v, that Av is also a probability vector, then A is a probability matrix.

PART II. If you are taking the class final, complete **THREE** additional questions for the prelim.

1. Suppose that A is an invertible square matrix. Prove that there exists numbers, $c_0, c_1, \ldots, c_{n-1}$ such that
$$A^{-1} = c_0 I + c_1 A + c_2 A^2 + \ldots + c_{n-1} A^{n-1}$$

2. Let V be the vector space of real polynomials of degree less than or equal to 2, that is, $V = \{a + bt + ct^2 : a, b, c \in R.\}$. Define the three functions from V into R by
$$\sigma_1(f(t)) = f(0), \quad \sigma_2(f(t)) = \int_0^1 f(t)dt, \quad \sigma_3(t) = f'(1).$$
Prove that this is a basis for the dual space, V^* and find a basis for V which is dual to $\{\sigma_1, \sigma_2, \sigma_3\}$.

3. Let
$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}$$
Verify that A and B commute and find a nonsingular matrix, P which diagonalizes both of these matrices. Why can commuting matrices with distinct eigenvalues always be simultaneously diagonalized?

4. Let A be a square matrix over C with minimum polynomial,
$$x^2(x-2)(x^2+2x+2)^2$$
and characteristic polynomial
$$x^2(x-2)^2(x^2+2x+2)^3$$
   (a) What is the Jordan canonical form for A
   (b) What is the real Jordan form?
   (c) What are the trace and the determinant of A?
   (d) Does the nullspace of A intersect the range of A?

5. Let V be the space of $n \times n$ matrices over C and fix a matrix, A.
   (a) Prove that set $V_0(A)$ of matrices in V commuting with A is a subspace.
   (b) Prove that if $B \in V_0(A)$ is invertible, then $B^{-1} \in V_0(A)$
   (c) Suppose A is symmetric. Then prove that if $B \in V_0(A)$ then so is B^T.
   (d) Suppose that A is diagonal with n *distinct* entries. Then prove that $B \in V_0(A)$ if and only if B is diagonal.

6. Let T be a linear map on a finite dimensional vector space, V such that for any subspace W, T maps W itno itself. (That is, W is invariant with respect to T.) Prove that T must then be a scalar multiple of the identity.