Ph.D. PRELIMINARY EXAMINATION
(supplement)

PART II – ANALYSIS

April 25, 2002

1. Answer any 2 of the 4 questions. Be sure to justify your reasoning.

2. Indicate below which 2 questions you wish to have graded. Do not indicate more than 2 questions for grading. Each question will be graded out of 10 points.

3. Use of soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.

4. Put your code number, but not your name, on each answer sheet that you submit. Confine your answers to the rectangular area indicated on the answer sheets.

5. Total exam time is one hour.

CODE NUMBER: ________________

GRADE QUESTIONS: 1. _________ 2. _________

3. _________ 4. _________

1. Let A be a compact subset of a metric space (M, d) and fix $y \in M \setminus A$. Prove that there exists $x^* \in A$ such that $\inf \{d(x, y) : x \in A\} = d(x^*, y) > 0$.

2. a) (5 points) Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers. Suppose that for each n, the following three inequalities hold: $x_n \leq y_n$, $x_n \leq x_{n+1}$, and $y_{n+1} \leq y_n$. Prove that $\{x_n\}$ converges.

   b) (5 points) Suppose that $\{a_n\}$ is a sequence of real numbers such that $a_n \geq 0$ for each n and $a_n \to 0$ as $n \to \infty$. Given any $\epsilon > 0$, prove that there exists a subsequence $\{b_n\}$ of $\{a_n\}$ such that $\sum_{n=1}^{\infty} b_n < \epsilon$.

3. Prove that

$$f(x) = \sum_{n=1}^{\infty} \frac{n^x}{2^n}$$

defines a continuous function on $\mathbb{R}$.

4. a) (6 points) Suppose the functions $\{f_n(x) : n = 1, 2, 3, \ldots\}$ are integrable and uniformly bounded on $[a, b] \subset \mathbb{R}$. For each n, let $F_n(x) = \int_a^x f_n(t)\,dt$, for $x \in [a, b]$. Show that there exists a subsequence F_{n_k} of F_n which converges uniformly on $[a, b]$.

   b) (4 points) Evaluate

$$\sum_{n=0}^{\infty} \frac{n+1}{2^n}.$$

Math 1540 Final Exam, April 25, 2002; 10 points per question
Identification Number:

1. Let $f(x,y) = (\sin(x+y), x-y+e^{xy})$.

   a) (4 points) Prove that f is locally invertible near $(0,0)$.

   b) (2 points) Let f^{-1} denote the local inverse of f shown to exist in a). Compute the derivative $Df^{-1}(0,1)$.

   c) (4 points) Suppose $g : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $||g(x,y)|| \leq M||(x,y)||^2$ for all $(x,y) \in \mathbb{R}^2$. Let $h = f + g$. Prove that $Dh(0,0) = Df(0,0)$.

2. Let f, g be C^1 functions from $\mathbb{R}^2$ to $\mathbb{R}$. Suppose that there exist a curve $S \subset \mathbb{R}^2$ and a constant real number c such that for all $(x,y) \in S$, both $g(x,y) = c$ and $g_y(x,y) \neq 0$ hold. Further, suppose that $\max\{f(x,y) : (x,y) \in S\}$ occurs at $(x_0, y_0) \in S$. Derive a formula for $f_x(x_0,y_0)$ in terms of $f_y(x_0,y_0), g_x(x_0,y_0)$, and $g_y(x_0,y_0)$.

3. Evaluate the following quantities.

   a) (5 points)
$$\lim_{n\to\infty} \int_0^1 \frac{1+x}{1+e^{-nx}}\, dx\,.$$

   b) (5 points)
$$\int_0^9 \int_{\sqrt{y}}^3 \cos(x^3)\, dx dy\,.$$

4. Let $f : [a,\infty[\to \mathbb{R}$ be Riemann integrable on each bounded interval in $\mathbb{R}$. Prove that if $\int_a^\infty f(x)\, dx$ exists and is finite, then for every $\epsilon > 0$, there exists $T > 0$ such that if $t_1, t_2 \geq T$, then $|\int_{t_1}^{t_2} f(x)\, dx| < \epsilon$.

5. Let V be an open, nonempty, bounded subset of $\mathbb{R}^2$ and let $u : V \to \mathbb{R}$ be a C^2 function.

   a) (4 points) Let $B_\delta(x,y)$ denote the ball of radius δ centered at (x,y) in $\mathbb{R}^2$. Suppose $\int\limits_{B_\delta(x_0,y_0)} u = 0$ for all $(x_0,y_0) \in V$ and $\delta > 0$ such that $B_\delta(x_0,y_0) \subset V$. Prove that $u = 0$ on V.

   b) (6 points) Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at all $(x,y) \in V$ if and only if $\int\limits_{\partial E} (\frac{\partial u}{\partial x}\, dy - \frac{\partial u}{\partial y}\, dx) = 0$ for all regions $E \subset V$ such that the topological boundary ∂E is a piecewise smooth curve.

6. Consider the three surfaces $S_1 = \{(x,y,z) : x^2 - y^2 + 2z^2 = 4\}, S_2 = \{(x,y,z) : y = 2\}$, and $S_3 = \{(x,y,z) : y = 0\}$ in $\mathbb{R}^3$, and let R denote the region bounded by S_1, S_2, and S_3.

   a) (3 points) Let C_1 denote the curve formed by $S_1 \cap S_2$ and let C_2 denote the curve formed by $S_1 \cap S_3$. Parametrize these curves with positive orientation relative to the outward normal vector to R.

   b) (7 points) Evaluate $\int_{C_1} F \cdot T\, ds$ for $F(x,y,z) = (xe^y, x + e^y, y(x + \cos z))$.