Ph.D. PRELIMINARY EXAMINATION
(supplement)

PART I – LINEAR ALGEBRA

April 26, 2001

1. Answer any 2 of the 10 questions.
2. Indicate below which 2 questions you wish to have graded. Do not indicate more than 2 questions for grading.
3. Use of soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.
4. Put your code number, but not your name, on each answer sheet that you submit. Confine your answers to the rectangular area indicated on the answer sheets.
5. Total exam time is one hour.

CODE NUMBER: ________________

GRADE QUESTIONS: 1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________ 7. ________

8. ________ 9. ________ 10. ________

LINEAR ALGEBRA Spring 2001

PRELIMINARY EXAMINATION SUPPLEMENT Time: 60 minutes.

INSTRUCTIONS

Remember to justify your answers. You ALWAYS have to prove your assertions. Please work any two problems completely. Each problem is worth 10 points.

Please mark clearly the problems which you submit.

(1) Let $M_n(\mathbb{F})$ denote the vector space of $n \times n$ matrices over a field $\mathbb{F}$, with the usual matrix addition and scalar multiplication, and for $A = (a_{ij}) \in M_n(\mathbb{F})$ let $tr(A) = \sum_i a_{ii}$.

(i) Show that $A \mapsto tr(A)$ defines a linear functional on $M_n(\mathbb{F})$ and that $tr(AB) = tr(BA)$ for all $A, B \in M_n(\mathbb{F})$.

(ii) Show also that for any $B = (b_{ij}) \in M_n(\mathbb{F})$ the formula $f_B(A) = tr(B^T A)$ defines a linear functional on $M_n(\mathbb{F})$ such that $f_B(E_{ij}) = b_{ij}$, where B^T is the transpose of B and E_{ij} is the matrix with a 1 in the (i, j) position and 0 elsewhere.

(iii) Show that any linear functional on $M_n(\mathbb{F})$ is of the form f_B for a unique $B \in M_n(\mathbb{F})$.

(2) Let V be a finite dimensional real vector space. Let $P: V \to V$ be a linear transformation. Denote the null-space of P by $V_1 = ker(P)$ and the null-space of $I_V - P$ by $V_2 = ker(I_V - P)$, where I_V is the identity transformation of V. Suppose $V = V_1 \oplus V_2$. Prove that $P^2 = P$.

Define the *dual space*, V^*, and the *dual transformation*, P^*. Show that $(P^*)^2 = P^*$. Hence show that $V^* = U_1 \oplus U_2$ where $U_1 = ker(P^*)$ and $U_2 = ker(I_{V^*} - P^*)$.

(3) Let V be a finite dimensional complex vector space, and let $T: V \to V$ be a linear transformation on V. Define the *minimal polynomial*, $m_T(x)$, and the *characteristic polynomial*, $c_T(x)$.

State the Cayley–Hamilton theorem, and prove that λ is a root of the minimal polynomial of T if and only if it is a root of the characteristic polynomial of T.

Give an example of a linear transformation $T : \mathbb{C}^4 \to \mathbb{C}^4$ with characteristic polynomial $c_T(x) = x^4 - x^3$ and minimal polynomial $m_T(x) = x^3 - x^2$. Is T diagonalizable?

(4) Let V be a real vector space of dimension n, with an inner product $\langle \cdot, \cdot \rangle$. Write $\|x\|$ for $\langle x, x \rangle^{1/2}$. Let $\{v_1, \ldots, v_n\}$ be orthonormal elements of V.

(i) Take any $x \in V$. Let $a_i = \langle x, v_i \rangle$, for $i = 1, \ldots, n$, and $s = \sum_{i=1}^{n} a_i v_i$. Show that for arbitrary real numbers $b_1, \ldots, b_n$:

$$\|x - s\| \leq \|x - t\| \qquad \text{where} \qquad t = \sum_{i=1}^{n} b_i v_i,$$

with equality if and only if $a_i = b_i$ for each $i = 1, \ldots, n$.

(ii) Show that $\{v_1, \ldots, v_n\}$ are linearly independent.

(iii) Deduce from (i) and (ii), that $x = s$.

MATH 2371. `Matrices and Linear Operators II` Spring 2001
`FINAL EXAMINATION` `Time: 120 minutes.`

INSTRUCTIONS

Remember to justify your answers. You ALWAYS have to prove your assertions except for the true-false questions. Please work

problem 1,
problem 2 or problem 3
problem 4 or problem 5
two problems out of problems 6, 7, 8 and 9.
one problem out of problem 10, 11 and 12,

for a total of *six* problems with maximum grade of 42 points. If you finish an allowable set of five problems, work the two problems from the trio 10, 11 and 12 which you did not choose earlier, for possible extra credit.

Please mark clearly the problems which you submit.

1. (8 points) True–False questions:

   (a) Let A and B be two $n \times n$ positive definite matrices. Then their regular matrix product $A.B$ is also positive definite.

   (b) Let N be a nilpotent $n \times n$ real matrix. Then $I + N$ has a square root.

   (c) A square matrix is triangulable if and only if its minimal polynomial is a product of linear factors.

   (d) Every linear operator has at least one cyclic vector.

   (e) Let $p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1} + x^n$ be a monic polynomial. Denote by P its companion matrix. Then we always have $\det(P) = c_0$.

   (f) Let V be vector space endowed with two inner products $\langle\ ,\ \rangle_0$ and $\langle\ ,\ \rangle_1$. Suppose that for every vector $v \in V$ we have

   $$\langle v, v\rangle_0 = \langle v, v\rangle_1.$$

   Then we have

   $$\langle v. w\rangle_0 = \langle v, w\rangle_1$$

   for every pair of vectors $v \in V$ and $w \in V$

   (g) Let $S \subset V$ be a subspace of a finite dimensional inner product space. The orthogonal complement of the orthogonal complement of S equals S.

   (h) A square matrix A is never similar to $A + I$.

2. (5 points) Let V be a finite-dimensional vector space, and let T be a linear operator on V such that $\text{rank}(T) = 1$. Prove that either T is diagonalizable or T is nilpotent, but not both.

3. (5 points) Let N_1 and N_2 be 3×3 nilpotent matrices. Prove that N_1 and N_2 are similar if and only they have the same minimal polynomial (You may use the cyclic decomposition theorem if you wish).

4. (5 points) Given the polynomial $p(x) = x^2(x-1)$, write down a matrix A whose minimal polynomial is $p(x)$.

5. (5 points) Prove that every positive matrix is the square of a positive matrix.

6. (7 points) Let T be a linear operator on an n-dimensional space, and suppose that T has n distinct characteristic values. Prove that any linear operator which commutes with T is a polynomial in T.

7. (7 points) Let V b a finite-dimensional inner product space and T a linear operator on V. Show that the range of T^* is the orthogonal complement of the null space of T.

8. (7 points) If T is a normal operator, prove that eigenvectors for T associated with distinct eigenvalues are orthogonal.

9. (7 points) Let V be the real inner product space consisting of the space of real-valued continuous functions on the interval $-2 \leq t \leq 2$, with the inner product

$$\langle f, g \rangle = \int_{-2}^{2} f(s)g(s)\, ds.$$

Let W be the subspace of odd functions, i.e., functions satisfying $f(-s) = -f(s)$. Find the orthogonal complement of W.

10. (10 points) Let T be linear operator in $\mathbb{R}^3$ which is represented by the matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

in the standard ordered basis. Find the matrices of a diagonalizable operator D and a nilpotent operator N in the standard ordered basis such that $T = D + N$ and $DN = ND$.

11. (10 points) Let A, B, C and D be $n \times n$ complex matrices which commute. Let E be the $2n \times 2n$ matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Prove that $\det(E) = \det(AD - BC)$.

12. (10 points) Let V be an n-dimensional complex vector space. Consider an operator $T: V \to V$ with distinct eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_k,$$

where $k \leq n$. Prove that $T^m(\alpha) \to 0$ as $m \to \infty$ for all $\alpha \in V$ if and only if $|\lambda_i| < 1$ for $i = 1, 2, \ldots, k$.