Ph.D. PRELIMINARY EXAMINATION
(supplement)

PART II – ANALYSIS

April 28, 2001

1. Answer any 2 of the 4 questions.
2. Indicate below which 2 questions you wish to have graded. Do not indicate more than 2 questions for grading.
3. Use of soft lead (#2) pencil or a dark ink pen to record your answers on the answer sheets that have been provided.
4. Put your code number, but not your name, on each answer sheet that you submit. Confine your answers to the rectangular area indicated on the answer sheets.
5. Total exam time is one hour.

CODE NUMBER: ________________

GRADE QUESTIONS: 1. _________ 2. _________

3. _________ 4. _________

1. Show that
$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+(x-n)^2}$$
defines a continuous function on $\mathbb{R}$.

2. Let f and g be continuous, real valued functions on a compact set K in $\mathbb{R}^n$ such that the maximum value of f occurs at an interior point p of K. Show that for every sufficiently small $\varepsilon > 0$, the function $f(x) + \varepsilon g(x)$ has a maximum at an interior point of K.

3. Give either a proof or a counterexample of each of the following statements.

   (a) If I_j is an open interval for each $j \in \mathbb{N}$, then $\cap_{j=1}^{\infty} I_j$ is an open subset of $\mathbb{R}$.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then for any bounded subset E of $\mathbb{R}$, the set $f(E)$ is also bounded.

   (c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then for any closed subset E of $\mathbb{R}$, the set $f(E)$ is also closed.

4. Let f and g be continuous function on the interval $[0, 1]$ with the property that
$$\int_0^1 x^n f(x)\, dx = \int_0^1 x^n g(x)\, dx$$
for every non-negative integer n. Show that $f = g$.

Math 1540 Final Exam
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Name or Identification number:

1. Let f be the mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$f(x,y) = (x^2 - y^2, 2xy).$$

   (a) Find the derivative $f'(1,1)$.

   (b) Let g be a local inverse of f satisfying $g(2,0) = (1,1)$. Find the derivative $g'(2,0)$.

2. Let $f(x,y)$ be a differentiable function on $\mathbb{R}^2$. Find a formula for $\frac{d}{dt}f(t,t^2)\big|_{t=1}$ in terms of the partial derivatives $\frac{\partial f}{\partial x}(1,1)$ and $\frac{\partial f}{\partial y}(1,1)$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function with the property that $f'(x,y)$ is invertible for every $(x,y) \in \mathbb{R}^2$. Show that for every open subset U of $\mathbb{R}^2$, the set $f(U)$ is also open.

4. Evaluate $\int_P e^{x+y}\,dx\,dy$ where P is the parallelogram with vertices $(0,0)$, $(-1,2)$, $(2,3)$, and $(1,5)$.

5. (a) Define the terms *closed form* and *exact form*.

   (b) Show that every continuous 1-form on $\mathbb{R}$ is exact.

   (c) Let ω be a closed k form on a subset E of R^n, and let $\Phi : B \to E$ be continuously differentiable, where B is the unit ball in $\mathbb{R}^m$. Show that $\eta = \Phi^*\omega$ is exact on B.

6. Let S and B denote the unit sphere and unit ball respectively in $\mathbb{R}^n$. The area of S is

$$\sigma_n = \int_S \sum (-1)^{i-1} x_i \, dx_i.$$

   Use Stokes' Theorem to express the volume of B in terms of σ_n.