Preliminary Exam in Analysis, January 6, 2024

Problem 1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$ and satisfies

$$\lim_{x \to 0} \frac{f(3x) - f(x)}{x} = \lambda,$$

where $\lambda \in \mathbb{R}$. Prove that f is differentiable at 0 and find $f'(0)$.
Hint: Consider the points $3^{-n}x$.

Problem 2. Suppose that a sequence of continuous functions $f_n : [-1, 1] \to \mathbb{R}$ satisfies:

(1) $f_n(x) \geq 0$ for $x \in [-1, 1]$ and $\displaystyle\int_{-1}^{1} f_n(x)\, dx = 1$;
(2) for any $c \in (0, 1)$, $\{f_n\}$ converges uniformly to 0 on $[-1, -c] \cup [c, 1]$.

Prove that for any continuous function $g \in C([-1, 1])$,

$$\lim_{n \to \infty} \int_{-1}^{1} g(x) f_n(x)\, dx = g(0).$$

Problem 3. Suppose that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous and $M > 0$ is a constant. Prove that if for every $x \in \mathbb{R}^n$ there is a unique $y = y(x) \in \mathbb{R}^m$ such that

$$|y(x)| \leq M \quad \text{and} \quad F(x, y(x)) = 0,$$

then the function $\mathbb{R}^n \ni x \mapsto y(x) \in \mathbb{R}^m$ is continuous.

Problem 4. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function whose partial derivatives of order ≤ 3 are everywhere defined and continuous. Let $\alpha = (1, \ldots, 1) \in \mathbb{R}^m$ and let $\mathbf{0}$ denote the origin in $\mathbb{R}^m$. Prove that

$$\sum_{n=1}^{\infty} \left[nf\Big(\frac{\alpha}{n}\Big) - nf\Big(-\frac{\alpha}{n}\Big) - 2\Big(\sum_{j=1}^{m} \frac{\partial f(\mathbf{0})}{\partial x_j}\Big) \right]$$

is a convergent series.

Problem 5. Let $f \in C^1(\mathbb{R})$ be a continuously differentiable function such that $|f'(x)| \leq 1/2$ for all $x \in \mathbb{R}$. Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x, y) = (x + f(y), y + f(x)).$$

Prove that

(1) g is a diffeomorphism,
(2) $g(\mathbb{R}^2) = \mathbb{R}^2$,
(3) the area $|g([0, 1]^2)|$ of the image of the unit square belongs to the interval $[3/4, 5/4]$.

Hint: Among other tools use the contraction principle.

Problem 6. Let $\mathbf{F}$ be a vector field in $U = \{x \in \mathbb{R}^3 : 1 < |x| < 2\}$, where $|x|$ is the Euclidean norm. Assume that there is a continuous function $f : (1, 2) \to \mathbb{R}$ such that

$$\mathbf{F} = f(|x|)x \quad \text{for all } x \in U.$$

Prove that if γ is a smooth closed curve in U, then

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{r} = 0.$$