Linear Algebra Preliminary Exam
January 7, 2024

1. Let A be an $n \times n$ complex matrix. Show that
$$\operatorname{rank} A + \operatorname{rank}(I - A) = n$$
if and only if $A^2 = A$.

2. Let $n \geq 2$ and P_n denote the vector space of complex polynomials $p(x)$ of degree less than n. Given $a = (a_1, \cdots, a_n)^T \in \mathbb{C}^n$, $T_a : P_n \to \mathbb{C}^n$ is the linear transformation defined by
$$T_a(p) = (p(a_1), p(a_2), \cdots, p(a_n))^T \in \mathbb{C}^n$$
for any $p \in P_n$. Under the basis $\{1, x, \cdots, x^{n-1}\}$ of P_n and the standard basis of $\mathbb{C}^n$, T_a is represented by an $n \times n$ complex matrix
$$T : \mathbb{C}^n \to \mathbb{C}^n.$$
(a) Find the entries T_{ij} of matrix T.
(b) Compute $\det T$ in terms of the $a_1, \cdots, a_n$.

3. Let X be the set of all $n \times n$ complex matrices and $A \in X$ be diagonalizable. Define $T_A : X \to X$ so that
$$T_A(B) = AB.$$
Show that T_A is also diagonalizable.

4. Let $A_i : \mathbb{C}^n \to \mathbb{C}^n$, $1 \leq i \leq k$ be k orthogonal projections satisfying for any $x \in \mathbb{C}^n$,
$$\sum_{i=1}^{k} \|A_i x\|^2 = \|x\|^2.$$
Show that
$$\sum_{i=1}^{k} A_i = I.$$

5. Let A be an $n \times n$ complex matrix preserving the orthogonal properties in $\mathbb{C}^n$, i.e., for any $x, y \in \mathbb{C}^n$, $(x, y) = 0$ if and only if $(Ax, Ay) = 0$. Show that $A = kU$ where k is a nonnegative number and U is a unitary matrix.

6. Let A, B be two $n \times n$ positive definite matrices. Show that
$$\det(A + B) \geq \det A + \det B.$$
(Hint: Prove first the special case when $A = I$.)