Linear Algebra Preliminary Exam
August 23, 2023

1. Let λ_1, λ_2 be two distinct eigenvalues of a complex square matrix A. Suppose that u is an eigenvector of A w.r.t. λ_1 and v is a generalized eigenvector of A^* w.r.t. $\overline{\lambda_2}$. Show that u, v are perpendicular to each other.

2. Let A be a 2×2 complex matrix. Show that the linear map

$$B \longmapsto AB + BA$$

is a bijection if and only if both $\operatorname{tr} A$ and $\det A$ are nonzero.

3. Let A be an $n \times n$ real positive definite matrix, $x \in \mathbb{R}^n$ be a nonzero real column vector, show that the determinant of the $(n+1) \times (n+1)$ matrix

$$B = \begin{pmatrix} A & x \\ x^T & 0 \end{pmatrix}$$

is negative.

4. Suppose A is an $n \times n$ complex matrix with only one Jordan block. Let B be an $n \times n$ matrix that commutes with A. Show that $B = f(A)$ for a polynomial f with complex coefficients.

5. Suppose $\{A_j\}_{j=1}^{n+1}$ are pairwise commuting $n \times n$ complex matrices. Suppose $A_1 A_2 \cdots A_{n+1} = 0$, show that in the above expression at least one of the factors can be removed with the expression still being zero.

6. Suppose two complex square matrices A, B satisfy $\|A - B\| = \|A\| - \|B\|$. Show that A^*A and B^*B share at least one eigenvector. Here $\|\cdot\|$ is the standard operator norm.