University of Pittsburgh
Department of Mathematics

Linear algebra preliminary exam

August 2017

Rule: You can use theorems proved in class or in the textbooks (provide proper reference), but if you use a statement from homework or tests you need to provide a proof.

Some notations: Given any linear transformation T, its transpose is denoted by T^t, and its adjoint with respect to an identified inner product is represented by T^*. Similarly, A^t denotes the transpose of a matrix A, while A^* denotes its adjoint, i.e. its conjugate transpose. The classical adjoint (or adjugate) of a square matrix is the transpose of the cofactor matrix $\mathrm{cof}(A)$ and is denoted by $\mathrm{adj}(A)$. A square complex matrix is said to be positive (resp. nonnegative), when A is Hermitian (self-adjoint) and for all nonzero $X \in \mathbb{C}^n$, the standard inner product $(AX, X) > 0$ (resp. nonnegative). The Hilbert-Schmidt inner product of two complex matrices $A, B \in \mathbb{C}^{n\times m}$ is defined by

$$A : B \stackrel{\text{def}}{=} \mathrm{Tr}(AB^*).$$

Problem 1. Let $A \in \mathbb{C}^{n\times n}$ be an arbitrary matrix.

(a) Prove that for all integer $m \geq 0$, $\mathrm{cof}(A^m) = \mathrm{cof(A)^m}$.

(b) Prove that if $n = 2$, $\mathrm{cof}(e^A) = e^{\mathrm{cof(A)}}$, where for any matrix $A \in \mathbb{C}^{n\times n}$, e^A is defined by

$$e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!}.$$

Problem 2. Let A and B be two Hermitian complex matrices.

(a) Prove that $\mathrm{Tr}(AB)$ is real.

(b) Prove that if A, B are positive, then $\mathrm{Tr}(AB) > 0$.

Problem 3. Let A be an $n \times n$ matrix with n distinct nonzero eigenvalues. Consider the linear map $S_A : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ given by:

$$S_A(X) = AX - XA.$$

(a) What are the dimensions of the null space and range of S_A?

(b) What are the eigenvalues and eigenvectors of S_A? Is S_A diagonalizable?

(c) What is the minimal polynomial of S_A when $A = \mathrm{diag}(1, 2, 3)$?

Problem 4. Let V be a real inner product vector space of finite dimension and let $W \subset V$ be a linear subspace. Assume that S is a positive operator on V. Let P be the orthogonal projection on W. Prove that PS is diagonalizable.

Problem 5. Let $R \in O(4)$.

(a) Prove that R admits an invariant two dimensional subspace $W \subset \mathbb{R}^4$.

(b) Prove that if $\det R > 0$, then it is similar to a matrix of the form

$$\begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & -d & c \end{bmatrix} \quad \text{with } a, b, c, d \in \mathbb{R}, \text{ and } a^2 + b^2 = c^2 + d^2 = 1.$$

Problem 6. Prove that for all n, Span $U(n) = \mathbb{C}^{n\times n}$.

Hint: Polar decomposition theorem could be useful.