PRELIMINARY EXAMINATION IN ANALYSIS
AUGUST 18, 2017

Problem 1. For any real numbers x and y, define

$$I_y(x) = \int_{\pi/2}^{\pi} \left(e^{-t^2 x} \cos(ty) - e^{t^2 y} \sin(tx) \right) dt$$

Prove that there exists $\epsilon > 0$ such that for each $y \in (-\epsilon, \epsilon)$, there is some $x \in (0,1)$ (depending on y) with $I_y(x) = 0$.

Problem 2. For any closed subset X of $\mathbb{R}^n$, prove that there is a countable subset S of X such that every continuous function $f\colon X \to \mathbb{R}$ is determined by its values on S.

Problem 3. Let $f(x) = \frac{\log x}{x}$, defined for any real number $x > 0$.

- Find the coefficients h_n of the Taylor series of f, centered at $x = 1$:

$$\mathcal{T}(f)(x) = \sum_{n=0}^{\infty} h_n (x-1)^n;$$

  and determine, with proof, all open real intervals on which the series $\mathcal{T}(f)$ converges uniformly to f.

- Does $\mathcal{T}(f)(2) = \sum_{n=0}^{\infty} h_n$ converge to $\frac{\log 2}{2}$? Explain.

Problem 4. Let $f(x) = \lfloor x \rfloor$ be the greatest integer function, defined on $\mathbb{R}$ by $f(x) = n$ for all $x \in [n, n+1)$, $n \in \mathbb{Z}$. Show that there is a sequence $(p_k)_{k \in \mathbb{N}}$ of polynomials converging pointwise to f on $\mathbb{R} - \mathbb{Z}$, such that for any compact set $K \subset \mathbb{R}$,

$$\int_K |f(x) - p_k(x)|\, dx \to 0 \quad \text{as } k \to \infty.$$

Problem 5. For a fixed $k \in \mathbb{N}$, define $f_k\colon \mathbb{R}^2 \to \mathbb{R}$ by:

$$f_k(x,y) = \begin{cases} \dfrac{x^2(x+y^2)}{x^2 + y^{2k}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f_1 is not differentiable at $(0,0)$, but f_k is differentiable at $(0,0)$ for each $k \geq 2$. (*Hint: At some point it may help to separately consider the cases* $|x| \geq |y|^k$ *and* $|x| \leq |y|^k$.)

Problem 6. The *quaternionic square* of $\mathbf{x} = (x, y, z, w) \in \mathbb{R}^4$ is

$$s(\mathbf{x}) = (x^2 - y^2 - z^2 - w^2, 2xy, 2xz, 2xw).$$

For any $\mathbf{b} \in \mathbb{R}^4$ such that $|\mathbf{b}| \leq 1/16$, show that the equation

$$s(\mathbf{x}) - \mathbf{x} + \mathbf{b} = \mathbf{0}$$

has a unique solution $\mathbf{x}$ such that $|\mathbf{x}| \leq 1/8$.