University of Pittsburgh
Department of Mathematics

Linear algebra preliminary exam

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Rule: You can use theorems proved in class or in the textbooks (provide proper reference), but if you use a statement from homework or tests you need to provide a proof.

Some notations: Given any linear transformation T, its transpose is denoted by T^t, and its adjoint with respect to an identified inner product is represented by T^*. Similarly, A^t denotes the transpose of a matrix A, while A^* denotes its adjoint, i.e. its conjugate transpose. The classical adjoint (or adjugate) of a square matrix is the transpose of the cofactor matrix $\mathrm{cof}(A)$ and is denoted by $\mathrm{adj}(A)$. A square complex matrix is said to be positive (resp. nonnegative), when A is Hermitian (self-adjoint) and for all nonzero $X \in \mathbb{C}^n$, the standard inner product $(AX, X) > 0$ (resp. nonnegative). The Hilbert-Schmidt inner product of two complex matrices $A, B \in \mathbb{C}^{n\times m}$ is defined by

$$A : B \stackrel{\text{def}}{=} \mathrm{Tr}(AB^*).$$

Problem 1. Assume that $A = [a_{jk}] \in \mathbb{C}^{n\times n}$ is a nonnegative rank 1 matrix. Prove that there exists a nonzero $v = (v_1, \cdots, v_n) \in \mathbb{C}^n$ such that

$$a_{jk} = v_j \bar{v}_k, \quad \forall j, k \in \{1, \cdots, n\}.$$

Problem 2. Let $O(n)$ be the set of orthogonal matrices in $\mathbb{R}^{n\times n}$. We define the set of special orthogonal matrices by

$$SO(n) := \{R \in O(n) \subset \mathbb{R}^{n\times n};\ \det R = 1\}.$$

(a) Let $W = \mathrm{Span}\, SO(2)$. Find a basis for W.

(b) Find $W^\perp$ with respect to the Hilbert-Schmidt inner product on $\mathbb{R}^{2\times 2}$.

(c) Prove that for any matrix $A \in \mathbb{R}^{2\times 2}$ there exist orthogonal matrices $R_1, R_2 \in O(2)$ and nonnegative scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$A = \alpha_1 R_1 + \alpha_2 R_2.$$

Problem 3. For a Hermitian matrix $A \in \mathbb{C}^{n\times n}$, let $\lambda_j(A)$ denote the j-th eigenvalue of A in increasing order, i.e. $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$. Let A and B be two Hermitian matrices. Use the min-max principle to prove that if $j + k > n$, then

$$\lambda_{j+k-n}(A+B) \leq \lambda_j(A) + \lambda_k(B).$$

Problem 4. Let V_n be the vector space of polynomials in $\mathbb{C}[x]$ of degree less than or equal n. Let the operator $T : V_n \to V_n$ be defined by $T(p) := p - p'$, where p' denotes the derivative of p.

(a) Prove that T is invertible and show that T^{-1} is a polynomial in T.

(b) Write the Jordan form of T^{-1} and justify your answer.

Problem 5. Assume that V is a finite dimensional inner product space and that $T \in L(V,V)$. Prove that T is normal iff there exists a unitary operator $U \in L(V,V)$ such that $T^* = UT$.

Remark: A proof that is valid for only complex vector spaces will not receive a full credit.

Problem 6. Suppose $A \in \mathbb{R}^{n\times n}$ is anti-symmetric, i.e. $A^t = -A$ and $B \in \mathbb{R}^{n\times n}$ is symmetric, i.e. $B^t = B$.

(a) Show that if n is even then $\text{cof(A)} : B = 0$.

(b) Prove that if n is even and B is of rank 1, then

$$\det(A+B) = \det A.$$

You can use that the determinant function is n-linear in matrix rows.