Preliminary Exam in Analysis, August 2023

Problem 1.

Let $E \subset \mathbb{R}^n$ be an open set and let $f : E \to \mathbb{R}$ be a C^1 function. Let $c \in E$ be such that $f(c) = 0$. Suppose that $df(c) \neq 0$. **Use the Inverse Function Theorem** to prove that there is an open set $U \subset \mathbb{R}^n$, an open set $V \ni c$, and a C^1 function $h : U \to V$ with a C^1 inverse $h^{-1} : V \to U$, such that

$$f(h(x_1, x_2, \ldots, x_n)) = x_n.$$

Problem 2.

Let $f \in C^\infty((-a, a))$ for some $a > 0$. Suppose that $f(0) = 0$ and $0 < f'(0) < 1$. Let

$$f_1(x) = f(x), \quad f_2(x) = f(f(x)), \quad \ldots, \quad f_n(x) = f(f_{n-1}(x)), \quad \ldots.$$

Prove that there exists $0 < \alpha \leq a$ such that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly in $[-\alpha, \alpha]$.

Problem 3

For all $\alpha \in A$ let $x_\alpha \in \mathbb{R}$, $x_\alpha \geq 0$. Define

$$\sum_{\alpha \in A} x_\alpha = \sup \left\{ \sum_{\alpha \in B} x_\alpha \colon B \subset A, B \text{ finite} \right\},$$

which is finite when the set of finite sums $\left\{ \sum_{\alpha \in B} x_\alpha \colon B \subset A, B \text{ finite} \right\}$ is bounded. Prove that in this case, there exist a countable subset $C \subset A$ such that $x_\alpha = 0$ when $\alpha \notin C$ and

$$\sum_{\alpha \in A} x_\alpha = \sum_{\alpha \in C} x_\alpha.$$

Problem 4

Given a positive integer n, prove that there is $\varepsilon > 0$ such that for every $n \times n$ matrix A with $|A| < \varepsilon$, there is an $n \times n$ matrix B satisfying $B^2 = A + B$. Here $|\cdot|$ can be any matrix norm. For example, if $A = [a_{ij}]_{i,j=1}^n$, one can take $|A| = \left(\sum a_{ij}^2 \right)^{1/2}$.

Problem 5.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and let $E \subset \mathbb{R}^n$ be a compact set. Assume that

1. $\Phi|_E : E \to \mathbb{R}^n$ is one-to-one.
2. For every $x \in E$, there is $r_x > 0$ such that $\Phi|_{B^n(x, r_x)} : B^n(x, r_x) \to \mathbb{R}^n$ is one-to-one.

Prove that there is an open set $U \subset \mathbb{R}^n$ such that $E \subset U$ and $\Phi|_U : U \to \mathbb{R}^n$ is one-to-one.

Problem 6

Let $f \colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ be one-to-one and of class C^1. Let $J(x) = \det Df(x)$ be its Jacobian determinant. For $x_0 \in \mathbb{R}^3$ let us denote by $Q_r(x_0)$ the cube centered at x_0 with side length r and edges parallels to the coordinate axes. Prove that

$$|J(x_0)| = \lim_{r \to 0^+} \frac{\operatorname{vol}(f(Q_r(x_0)))}{r^3} \leq \limsup_{x \to x_0} \frac{\|f(x) - f(x_0)\|^3}{\|x - x_0\|^3}.$$

Here $\|x\| = \sqrt{\sum_{i=1}^3 x_i^2}$ is the Euclidean norm in $\mathbb{R}^3$.