Linear Algebra Preliminary Exam
September 5 2020

Problem 1 *Let A, B be complex square matrices with minimal polynomials $m_A(s)$, $m_B(s)$ and characteristic polynomials $p_A(s)$, $p_B(s)$. Show that if*

$$m_A(s) = p_B(s) \text{ and } m_B(s) = p_A(s),$$

then A and B are similar.

Problem 2 *Let A_n be a $2^n \times 2^n$ real matrix defined recursively by*

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$A_n = \begin{pmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -A_{n-1} \end{pmatrix} \text{ for } n \geq 2.$$

Find $\det A_n$ and justify your answer. (Hint: Calculate A_n^2.)

Problem 3 *Let A be a real $n \times n$ matrix. Suppose that the symmetric matrix $A + A^T$ has eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. Show that any eigenvalue λ of A satisfies*

$$\frac{\mu_1}{2} \leq \operatorname{Re} \lambda \leq \frac{\mu_n}{2}.$$

Problem 4 *Let M_3 be the collection of all 3×3 complex matrices with the natural linear structure. Let $T : M_3 \to M_3$ be the linear map*

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21}+a_{12} & a_{13} & 0 \\ a_{31} & 0 & a_{13} \\ 0 & a_{31} & a_{32}+a_{23} \end{pmatrix}.$$

(a) Show that T is Nilpotent, i.e., $T^k = 0$ for some $k \geq 1$.
(b) Find the Jordan Canonical Form of T. (Hint: The JCF should be a 9×9 matrix.)

Problem 5 *Let A be an anti-selfadjoint map on a finite-dimensional complex Euclidean space. Show that*
(a) $A - I$ is invertible.
(b) If $U = (A+I)(A-I)^{-1}$, then U is unitary and $U - I$ is invertible.

Problem 6 *(a) Let $x = (x_1, \cdots, x_n)^T$, $y = (y_1, \cdots, y_n)^T$ be two column vectors in $\mathbb{R}^n$, $n \geq 2$, such that $|x| = |y| = 1$ and $x \perp y$. Show that*

$$x_1^2 + y_1^2 \leq 1.$$

(b) Let A be a real $n \times n$ symmetric matrix with n eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Show that for any $x, y \in \mathbb{R}^n$ such that $|x| = |y| = 1$ and $x \perp y$,

$$|(x, Ax) + (y, Ay)| \leq |\lambda_1| + |\lambda_2|.$$