Linear Algebra Preliminary Exam
January 7 2021

1. Let Y and Z be subspaces of a (not necessarily finite dimensional) linear space X.
(a). Show that $(Y+Z)^{\perp} = Y^{\perp} \cap Z^{\perp}$.
(b). If $X = Y \oplus Z$, then $X' = Y^{\perp} \oplus Z^{\perp}$, Z' is isomorphic to $Y^{\perp}$, and Y' is isomorphic to $Z^{\perp}$.
Here $Y^{\perp}$, $Z^{\perp}$ denote the annihilators of Y and Z respectively, and X', Y', Z' are the dual spaces of X, Y, Z.

2. Let X be the linear space of all 3×3 complex matrices. Let
$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
and $T : X \to X$ be the linear map such that for any $A \in X$
$$TA = AN - NA.$$
(a). Show that $T^k = 0$ for some integer $k \geq 1$ and find the smallest such k.
(b). Find the Jordan Canonical Form of T.

3. Let A be a complex $m \times n$ matrix and B be a complex $n \times m$ matrix where m, n are two positive integers. Show that $I - AB$ is invertible if and only if $I - BA$ is invertible.

4. Let A be a complex $n \times n$ matrix which commutes with all reflection matrices of the form
$$R = I - 2vv^T$$
where $v \in \mathbb{R}^n$ is a unit column vector. Show that $A = \lambda I$ for some scalar $\lambda \in \mathbb{C}$.

5. Let A be an $n \times n$ self-adjoint matrix such that for any $n \times n$ positive definite matrix B, the trace
$$\operatorname{tr} AB > 0.$$
Show that $A > 0$.

6. Let A be a square matrix such that
$$\left\|A^2\right\|_{HS} = \|A\|_{HS}^2 .$$
Show that A is normal. Here $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm of a matrix, i.e., for $A = (a_{ij})_{n\times n}$,
$$\|A\|_{HS} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} .$$