Linear Algebra Preliminary Exam
August 2014

Problem 1 *(12 points) Let Y be a subspace of a finite dimensional linear space X. Show that*

$$\dim X/Y = \dim X - \dim Y.$$

Problem 2 *(12 points) Let A be an $n \times n$ complex matrix such that $A^2 = A$ and $\mathrm{rank} A = r$. Prove that A is similar to* $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

Problem 3 *(12 points) Let A, B be two self-adjoint $n \times n$ matrices such that*

$$\min_{\lambda \in \sigma(A)} \lambda > \max_{\mu \in \sigma(B)} \mu.$$

Show that $A > B$. Here $\sigma(A)$ and $\sigma(B)$ are spectrums of A, B respectively.

Problem 4 *(12 points) Let $Q : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal map which is a $\pi/2$ angle rotation about the axis in the direction of vector $(1, 1, 1)$. This rotation follows the right hand rule with the thumb pointing to the v_1 direction. Find its matrix representation.*

Problem 5 *(12 points) Let A, B be two positive-definite self-adjoint $n \times n$ matrices. Suppose $AB = BA$. Prove that $AB > 0$.*

Problem 6 *(12 points) Let X be a finite dimensional linear space and let $P_1, P_2 : X \to X$ be two projections, i.e.,*

$$P_1^2 = P_1,\ P_2^2 = P_2.$$

Suppose that their ranges satisfy $R_{P_1} \subset R_{P_2}$, is it always true that $P_1 P_2 = P_2 P_1$. Justify your answer.