Linear Algebra Preliminary Exam
April 2014

Problem 1 *(12 points) Let X be a finite dimensional linear space and $T \in L(X, X)$. Suppose*

$$\dim(R_{T^2}) = \dim(R_T),$$

show that

$$R_T \cap N_T = \{0\}.$$

Here R_T is the range of T and N_T is the null space of T.

Problem 2 *(12 points) Let*

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Calculate

$$A^{2014}.$$

Problem 3 *(12 points) Let A, B be $n \times n$ complex matrices and $A > 0$. If there exists a $k \in \mathbb{N}$ such that $A^k B = BA^k$, prove that $AB = BA$.*

Problem 4 *(12 points) Let A be a Hermitian matrix and $B = \operatorname{Re} A$, the real part of A. Show that*

$$\max_{\mu \in \sigma(B)} \mu \leq \max_{\lambda \in \sigma(A)} \lambda.$$

Here $\sigma(A), \sigma(B)$ are the spectrums of A, B respectively.

Problem 5 *(12 points) Let $A, B : \mathbb{C}^n \to \mathbb{C}^n$ be two orthogonal projections satisfying for any $x \in \mathbb{C}^n$,*

$$\|Ax\|^2 + \|Bx\|^2 = \|x\|^2.$$

Show that

$$A + B = I.$$

Problem 6 *(12 points) Let $A = (a_{ij})$ be an $n \times n$ real matrix satisfying the following diagonally dominant condition*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad \text{for all } i = 1, 2, \ldots, n.$$

Prove that A is invertible.