PRELIMINARY EXAMINATION IN LINEAR ALGEBRA
MAY 3, 2019

You may use facts *proved* in class or in Hoffman/Kunze (provide a proper reference). Justify all other assertions, including those from homework or test problems.

Problem 1.

(a) Prove or give a counterexample: if $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $\operatorname{null}(T) \cap \operatorname{range}(T)$ has dimension at least 1 then T is nilpotent.

(b) Prove or give a counterexample: if $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation such that $\operatorname{null}(T) \cap \operatorname{range}(T)$ has dimension at least 2 then T is nilpotent.

Problem 2. Suppose that λ is an eigenvalue of a matrix $A \in \mathbb{C}^{n\times n}$ with algebraic multiplicity k. Show that $(A - \lambda I)^k$ has rank $n - k$.

Problem 3. For any $n \geq 1$, classify the matrices $Q \in \mathbb{R}^{n\times n}$ that are both orthogonal and skew-symmetric, meaning $Q^t = -Q$, up to similarity; i.e. exhibit exactly one representative from each real similarity class. (*Hint*: the answer is very different for odd versus even n.)

Problem 4.

(a) For a diagonalizable $n \times n$ matrix A, show that $\det(e^A) = e^{\operatorname{trace}(A)}$, where e^A is the matrix exponential of A: $e^A = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

(b) Now for an arbitrary 2×2 matrix A with trace equal to 0, show that $\det(e^A) = 1$. (Do not ignore the non-diagonalizable case!)

Problem 5.

(a) Show that if the self-adjoint part of a matrix A is positive-definite then A is invertible and the self-adjoint part of A^{-1} is positive-definite.

(b) Let a be a fixed positive real number. Show that if a self-adjoint matrix A is positive-definite then $\|W\| < 1$, where $W = (I - aA)(I + aA)^{-1}$ and $\|\cdot\|$ is the operator norm induced by the standard Euclidean norm.

Problem 6. Consider

$$B = \begin{pmatrix} L & M \\ O & N \end{pmatrix} \in \mathbb{C}^{2n\times 2n}$$

for $L, M, N, O \in \mathbb{C}^{n\times n}$ such that O is the zero matrix.

(a) Show that if B is diagonalizable, then L and M must be diagonalizable.

(b) Show that if L and M are diagonalizable and do not share eigenvalues, then B is diagonalizable.