Rule: You can use theorems proved in class or in the textbooks (provide proper reference), but if you use a statement from homework or tests you need to provide a proof.

Problem 1. Let $A \in \mathbb{R}^{n \times n}$ be a matrix whose components are either 1 or $-1$. Prove that $\det A = 2^{n-1}m$ where $m$ is an integer.

Problem 2. Assume that $V$ is a finite dimensional complex vector space. Suppose $T, U \in L(V, V)$ are two operators and that $TU - UT$ is nonnegative. Prove that $T$ and $U$ have a common eigenvector.

Problem 3. Prove that for any two matrices $A, B \in \mathbb{R}^{n \times n}$,
$$\det(I - AB) = \det(I - BA).$$

Problem 4.
(a) Assume $A \in \mathbb{C}^{n \times n}$ has $n$ distinct nonzero eigenvalues. Prove that there are exactly $2^n$ distinct matrices $B$ such that $B^2 = A$ (i.e., in particular, there are no more than $2^n$ matrices with this property).

(b) How many such matrices $B \in \mathbb{C}^{3 \times 3}$ exist if $A = \text{diag}(2, 2, 1)$. Why?
Problem 5. Let \( f: \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C} \) be the function defined for all \( x, y \in \mathbb{C}^{n+1} \) by
\[
f(x, y) := \sum_{j=1}^{n} x_j \bar{y}_j - x_{n+1} \bar{y}_{n+1}.
\]

(a) For \( A, B \in \mathbb{C}^{(n+1)\times(n+1)} \), show that \( f(Ax, y) = f(x, By) \) for all \( x, y \in \mathbb{C}^{n+1} \) if and only if \( A = JB^*J \), where \( J = \text{diag}(1, \ldots, 1, -1) \).

(b) We define
\[
U(n, 1) := \{ U \in \mathbb{C}^{(n+1)\times(n+1)} \mid f(Ux, Uy) = f(x, y) \ \forall x, y \in \mathbb{C}^{n+1} \}.
\]
Show that if \( U \in U(n, 1) \) then \( U \) is invertible. How do \( U^{-1} \) and \( U^* \) relate?

Problem 6. Prove that there exists a constant \( C > 0 \) such that for all matrices \( A \in \mathbb{C}^{3\times3} \), there exists \( P \in U(3) \) and a diagonal \( D \in \mathbb{C}^{3\times3} \) such that
\[
\|P^*AP - D\|^2 \leq C\|A^*A - AA^*\|,
\]
where \( \|B\| := \text{Tr}(BB^*)^{1/2} \) denotes the Hilbert-Schmidt norm of \( B \).

**Hint:** Prove the statement for the class of triangular matrices first.