University of Pittsburgh

Linear algebra preliminary exam

May 2016

You can use theorems proved in class, but if you use a statement from homework or tests you need to provide a proof.

Problem 1. Let A and B be $n \times n$ complex matrices. Show that AB and BA have the same characteristic polynomial. Is it true that AB and BA also have the same minimal polynomial? (Remember to consider the case when A and B are not invertible.)

Problem 2. Let A, B be $n \times n$ real matrices. Suppose $A + B = I$ and $\text{rank}(A) + \text{rank}(B) = n$ show that $R_A \cap R_B = \{0\}$. (Recall that R_A denotes the range of A or column space of A.)

Problem 3. Prove that any unitary matrix Q has a square root which is also unitary, i.e. there is a unitary matrix R such that $R^2 = Q$.

Problem 4. Let $v \in \mathbb{C}^n$ be a nonzero vector (written as a column vector) and let M be the $n \times n$ matrix defined by $M = vv^T$.

(a) Find the eigenvalues, eigenvectors, characteristic polynomial and minimal polynomial of M.

(b) Is M diagonalizable?

Problem 5. Let A be an $n \times n$ complex matrix such that the minimal polynomial m_A is $m_A(t) = t^n$. Prove that there is a vector $v \in \mathbb{C}^n$ such that $\{v, Av, \ldots, A^{n-1}v\}$ is a basis for $\mathbb{C}^n$.

Problem 6.

(a) Let A and B be $n \times n$ complex matrices which are non-negative (i.e. self-adjoint and all the eigenvalues are nonnegative). Show that:

$$\det(A + B) \geq \det(A) + \det(B).$$

(b) Show that the set of positive (definite) matrices with determinant > 1 is a convex set. Recall that a set S is convex if for any $A, B \in S$ and $0 \leq t \leq 1$ we have $tA + (1 - t)B \in S$. (Hint: log-concavity.)