University of Pittsburgh

Linear algebra preliminary exam

May 2016

You can use theorems proved in class, but if you use a statement from
homework or tests you need to provide a proof.

Problem 1. Let $A$ and $B$ be $n \times n$ complex matrices. Show that $AB$ and
$BA$ have the same characteristic polynomial. Is it true that $AB$ and $BA$
also have the same minimal polynomial? (Remember to consider the case
when $A$ and $B$ are not invertible.)

Problem 2. Let $A, B$ be $n \times n$ real matrices. Suppose $A + B = I$ and
rank($A$) + rank($B$) = $n$ show that $R_A \cap R_B = \{0\}$. (Recall that $R_A$
denotes the range of $A$ or column space of $A$.)

Problem 3. Prove that any unitary matrix $Q$ has a square root which is
also unitary, i.e. there is a unitary matrix $R$ such that $R^2 = Q$.

Problem 4. Let $v \in \mathbb{C}^n$ be a nonzero vector (written as a column vector)
and let $M$ be the $n \times n$ matrix defined by $M = vv^T$.

(a) Find the eigenvalues, eigenvectors, characteristic polynomial and min-
imal polynomial of $M$.

(b) Is $M$ diagonalizable?

Problem 5. Let $A$ be an $n \times n$ complex matrix such that the minimal
polynomial $m_A$ is $m_A(t) = t^n$. Prove that there is a vector $v \in \mathbb{C}^n$
such that $\{v, Av, \ldots, A^{n-1}v\}$ is a basis for $\mathbb{C}^n$.

Problem 6.

(a) Let $A$ and $B$ be $n \times n$ complex matrices which are non-negative (i.e.
self-adjoint and all the eigenvalues are nonnegative). Show that:

$$\det(A + B) \geq \det(A) + \det(B).$$

(b) Show that the set of positive (definite) matrices with determinant $> 1$
is a convex set. Recall that a set $S$ is convex if for any $A, B \in S$ and
$0 \leq t \leq 1$ we have $tA + (1 - t)B \in S$. (Hint: log-concavity.)