University of Pittsburgh

Linear algebra preliminary exam

August 2016

You can use theorems proved in class, but if you use a statement from homework or tests you need to provide a proof.

Problem 1. Consider the vector space $\mathrm{M}_{n\times n}(\mathbb{R})$ of $n \times n$ real matrices. Consider the linear map $T : \mathrm{M}_{n\times n}(\mathbb{R}) \to \mathrm{M}_{n\times n}(\mathbb{R})$ given by $T(A) = A^t$ for all $A \in \mathrm{M}_{n\times n}(\mathbb{R})$. Here A^t denotes the transpose of A.

(a) Find the characteristic polynomial and minimal polynomial of T.

(b) Find the Jordan form of T.

Problem 2. Let V be a finite dimensional vector space and $A : V \to V$ a linear map. Suppose $N(A) = N(A^2)$. Show that for any integer $m > 0$ we have $N(A^m) = N(A)$. (Here N denotes the null space.)

Problem 3. Let $\{v_1, v_2, v_3\} \subset \mathbb{Z}^3$ be vectors with integer coordinates. Show that every vector in $\mathbb{Z}^3$ is a linear combination of the v_i with integer coefficients if and only if the (Euclidean 3-dimensional) volume of the parallelepiped P they form is equal to 1. The parallelepiped P formed by the v_i is the set:

$$P = \{c_1v_1 + c_2v_2 + c_3v_3 \mid 0 \le c_i \le 1,\ \forall i = 1, 2, 3\}.$$

Problem 4. Prove that the set of diagonalizable matrices is dense in the set of all complex $n \times n$ matrices, that is, any matrix is a limit of a sequence of diagonalizable matrices.

Problem 5. Suppose $A = (a_{ij})$ is an $n \times n$ positive definite matrix over $\mathbb{C}$. Show that the determinant of any principal submatrix of A is a positive real number. (A principal submatrix is a submatrix of A whose entries are a_{pq} for $p, q \in I$, where $I = \{i_1, \ldots, i_k\}$ is some subset of $\{1, \ldots, n\}$.)

Problem 6. Let M be an $n \times n$ complex matrix. The nullity sequence of M is the sequence $(n_1(M), n_2(M), \ldots)$ where $n_k(M) = \mathrm{null}(M^k) := \dim(N(M^k))$ for every k. Let A, B be $n \times n$ complex nilpotent matrices. Show that A and B are similar if and only if they have the same nullity sequence.