SOME MATHEMATICAL PROBLEMS IN SHAPE FORMATION

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In mathematics, we formulate conjectures and discover principles attesting to the coherence and harmony in nature. What is specific to mathematics is that we achieve this through rigorous deduction. Therefore, mathematics may provide a firm ground to our empirical understanding of the physical phenomena. It is important, however, to remember that mathematicians (like myself) engage in pure mathematics, without having any concrete "application" in mind but in a free pursuit driven by intellectual curiosity. This curiosity is rewarded by surprising discoveries of the elegant and consistent structures in the abstract objects we study. Sometimes, it also leads to discoveries regarding the practical applications for what began as pure mathematics. Therefore, there is no clear line separating pure and applied mathematics.

Recently, there has been sustained interest in the growth-induced morphogenesis, particularly of the low-dimensional structures such as fillaments, laminae and their assemblies, arising routinely in biological systems and their artificial mimics. The physical basis for morphogenesis can be presented in terms of a simple principle: differential growth in a body leads to residual strains that generically result in changes of the body's shape. Eventually, the growth patterns are expected to be, in turn, regulated by these strains, so that this principle might well be the basis for the physical self-organization of the biological tissues.

While such questions lie at the interface of biology, chemistry and physics, fundamentally they have a deeply geometric and analytical character. Indeed, they may be seen as a variation on a classical theme in differential geometry - that of embedding a shape with a given metric in a space of possibly different dimension. The goal now, in addition to stating the conditions when it might be done (or not), is to: 1) constructively determine the resulting shapes in terms of an appropriate mathematical theory, and: 2) investigate the separation of scales which arise, naturally, in slender structures and discover the constraints associated with the prescription of a metric.

Some somewhat more precise statements. Let G be a smooth Riemannian metric, given on an open, bounded and simply connected domain $U \subset \mathbb{R}^3$. We consider the following functional, which can be interpreted as the "energy" of a deformation u of the reference state U:

$$E(u) = \int_U W\left((\nabla u)\sqrt{G}^{-1}\right)\,\mathrm{d}x \qquad \forall u \in W^{1,2}(U,\mathbb{R}^3). \tag{1}$$

Above, a vector field $u : U \to \mathbb{R}^3$ is assumed to belong to the space $W^{1,2}$, i.e. u has its first derivatives square integrable in its domain U. Such formalism is indeed relevant for the description of processes in plasticity, swelling and shrinkage in thin films, or plant morphogenesis. We can simply say that the model in (1) postulates that the 3-dimensional elastic body U seeks to realize a configuration with a prescribed Riemannian metric G.

Since $W(F) = 0$ if and only if $F \in SO(3)$ (i.e. when F is a rotation), it follows for $F = (\nabla u)\sqrt{G}^{-1}$ that this is the case if and only if $(\nabla u)^T\nabla u = G$ and $\det \nabla u > 0$. One uses the polar decomposition theorem, known to every good student of mathematics. Therefore, $E(u) = 0$ only when u is an orientation preserving isometric immersion of G into $\mathbb{R}^3$. Another classical theorem now states such immersion exists (and is automatically smooth) if and only if the Riemann curvature tensor R_G of G vanishes identically in U.

Observe that one could also define "the energy" as the difference between the pull-back metric of a deformation u and the given metric: $I(u) = \int |(\nabla u)^T \nabla u - G|^2 \, \mathrm{d}x$. But, another celebrated theorem states that there always exists Lipschitz continuous u with $I(u) = 0$. If $R_G(x_0) \neq 0$ then such u necessarily has a "folding structure", namely it cannot be orientation preserving (or reversing) in any open neighborhood of x_0. Indeed, we proved that the functional E, which only approves of the orientation preserving deformations, has strictly positive infimum for non-flat G:

$$R_G \not\equiv 0 \;\Leftrightarrow\; \inf\left\{E(u);\; u \in W^{1,2}(U, \mathbb{R}^3)\right\} > 0, \tag{2}$$

which justifies interpreting the quantity inf E as a measure of the residual stress at free equilibrium (i.e. in the absence of external forces or boundary conditions).

We now have a large poll of results related to the functional (1), for a variery of the Riemannian metrics G, given on the thin sheet as a function of location in the central plane and also across its thickness. The shape is then a consequence of the energy minimization in E on the frustrated geometrical object. Naturally, in the context of thin domains, this analysis can be seen as an extension of the derivation of a hierarchy of plate theories in nonlinear elasticity. This fruitful research direction has been furthered by deep results both in the case of plates and shells. There are also many interesting experimental results allowing for printing a metric G into a gellic disk, and then comparing their compatibility with our findings.

Conclusion and an advertisment! Our abstract results may be applied to the model energy of the nematic liquid crystal elastomers, which are rubber-like, cross-linked, polymeric solids, having both positional elasticity due to solid response of the polymer chains, and the orientation elasticity due to the separately deforming director. Another particular choice of the metric G is consistent with the experiment fabricating programmed flat disks of gels having a non-constant monomer concentration which induces a "differential shrinking factor". In this experiment, the disk is then activated in a temperature raised above a critical threshold, whereas the gel shrinks locally with a factor proportional to its concentration.

Similar questions can, naturally, also be posed on networks, via a discrete differential geometry approach; however, this area is much less developed than the well established field of classical differential geometry and the calculus of variations. Our forthcoming Math Research Center semester-long program at the University of Pittsburgh, will concentrate exactly on such topics. Following our previous successful programs (namely, the 2013 Semester on Game Theory and PDEs, and the 2014 Semester on Convex Integration and Analysis), we will host the "Fall 2014 Theme Semester on Discrete Networks: Geometry, Dynamics and Applications". This program, which is organized by B. Doiron, B. Ermentrout, J. Rubin and M. Lewicka, will be devoted to studying topics pertaining to: discrete networks, structural rigidity and morphogenesis of discrete structures, dynamics on networks, and dynamic network problems in neuroscience. It will take place in the period of September–December 2014.

Stay tuned for more information coming soon!