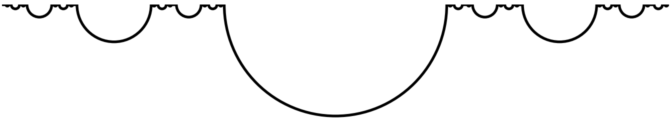
Intuitionistic logic, negative dimensional tensors, and angular momentum

Jonathan Holland

University of Pittsburgh

April 9, 2015



Abstract

In the 1970s, Roger Penrose proposed a kind of object that he called a "negative-dimensional tensor". These could be used to decompose the structure constants of the Lie algebra of angular momentum in three dimensions into a simple product of tensors of dimension -2. I shall give a construction of such negative-dimensional objects, that rests on an intuitionistic interpretation of the famous Euler "identity" $2 + 4 + 8 + 16 + ... = -2$.

Outline

Introduction and motivation

- Fundamental theorem of twistor theory (2013): The collection of zero rest-mass fields in space-time of all helicities forms a representation of the Heisenberg algebra in four degrees of freedom.
- Much of our work since then has been devoted to how to understand what such a result might mean in a curved space-time, especially at very high energies present at the moment of the big bang.
- Candidate geometries, harmonic analysis, etc
- What kinds of mathematics describe the universe?

Why intuitionism?

- The principle of parsimony in science: The “simplest” explanation is often the best. The “simplest” rules for inference are those based on constructive mathematics.
- The information paradox: Infinite quantities of information cannot be represented in any finite region of space-time. (Black holes, cosmic censorship, etc.)
- The reality/complexity problem: Why do real and complex numbers (apparently) describe the world so well?

Data streams

- A data stream is a sequence of bits, 0's and 1's
- We do not necessarily know in advance how large a data stream is, or even if it is finite (zero terminated) or infinite.
- How do we decide if two data streams are equal?
- We compare them, bit by bit, using a Turing machine that halts iff they are unequal, and outputs the index of the first bit at which the two data streams disagree.

Semantics of equality and inequality

- We will write $A =_n B$ if A and B agree up to n places.
- We shall think of this as assigning a *credence* to the statement "$A \neq B$". The credence of $A \neq B$, if A and B agree up to n places, is 2^{-n}.
- Credence defines a metric on the set of all binary sequences.

The 2-adic numbers

- The “intuitionistic” metric that we have described is closely related to the 2-adic metric in number theory.
- Any positive integer n is uniquely expressible as $n = k \cdot 2^m$ with k odd. The 2-adic norm is defined by

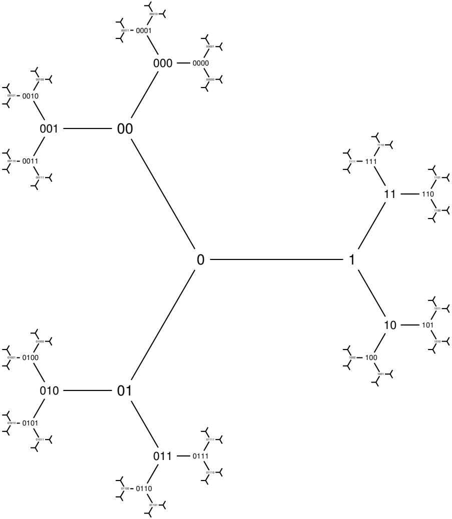
$$|n|_2 = 2^{-m}.$$

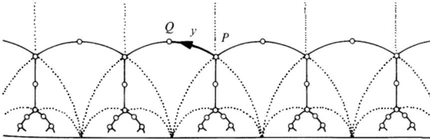
  The metric is $d(n_1, n_2) = |n_1 - n_2|_2$.
- This norm satisfies a strong version of the triangle inequality:

$$|n_1 + n_2|_2 \leq \max\{|n_1|_2, |n_2|_2\}.$$

  (The ultrametric inequality.)
- If we regard the positive integers in binary, then they are identified with the set of finite binary sequences. The completion of $\mathbb{N}$ under this metric is the set of infinite binary sequences, with the “intuitionistic” metric.

The binary tree, Cantor set, and $SL_2(\mathbb{Z})$

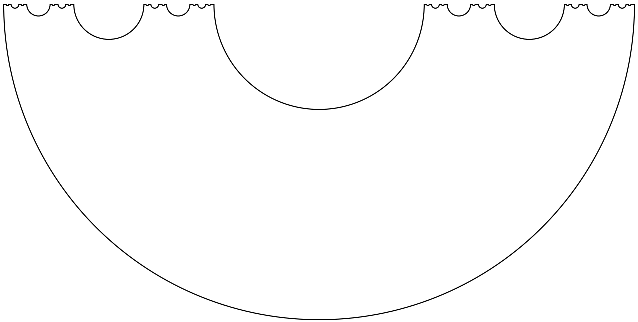




Reality

Theorem

Associated to any suitably generic $\{0, 1\}$-valued harmonic function on the binary tree is a natural S^1, as a real manifold, with a projective structure.



A curious identity

- $x + x^2 + x^3 + \cdots = \frac{x}{1-x}$ (provided $|x| < 1$)
- Evaluating this at $x = 2$ gives

  $$2 + 4 + 8 + 16 + \cdots = -2.$$

  (Euler)
- Obviously, we shouldn't take this too seriously.
- But in fact, this is true exactly as stated over the 2-adic integers!

A model in spins

- Let V be a two-dimensional complex vector space, spanned by two vectors which we denote $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$.
- Then $V \otimes V$ is the four dimensional space spanned by

$$
\begin{aligned}
|{\uparrow\uparrow}\rangle &= |{\uparrow}\rangle \otimes |{\uparrow}\rangle & \qquad |{\uparrow\downarrow}\rangle &= |{\uparrow}\rangle \otimes |{\downarrow}\rangle \\
|{\downarrow\uparrow}\rangle &= |{\downarrow}\rangle \otimes |{\uparrow}\rangle & \qquad |{\downarrow\downarrow}\rangle &= |{\downarrow}\rangle \otimes |{\downarrow}\rangle
\end{aligned}
$$

- Then $V \otimes V \otimes V$ is the eight-dimensional vector space spanned by all configurations of the spins of three particles into up and down states. Etc

Fock space

- Now let us form the “Fock space” by taking the direct sum of all of these different spins:

$$H = V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

- The dimension is $2 + 4 + 8 + 16 + \cdots$.
- Is there a legitimate sense in which this equals -2? (And why should we care?)

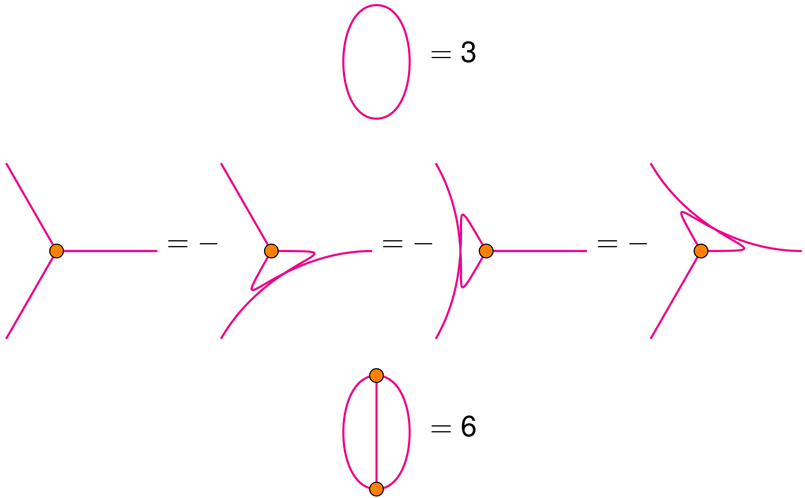
Observables

- Observables are self-adjoint projection operators.
- The dimension of the space of fixed points of a (finitely supported) observable gives an integer-valued function on the observables.
- The trace of the identity is the sum $2 + 4 + 8 + 16 + \cdots$

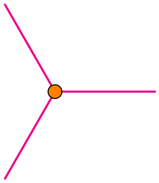
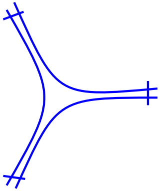
An intuitionistic model

- Let H be the space of finitely supported $F_2 = \{0, 1\}$-valued functions on the binary tree. (A vector space over F_2.)
- The trace of a finitely-supported endomorphism of H is defined as the number of fixed points of the endomorphism.
- The trace is integer valued, and is Lipschitz with respect to a pair of 2-adic metrics.
- It extends continuously to the completion. In the completion, we have $\mathrm{tr}(I) = -2$!

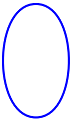
Angular momentum axioms (Penrose, 1971



Factorization into negative-dimensional "binors"

 $=$ 

where

 $= -2$

Thank you!