The Frobenius Coin Problem

A Geometric Proof for Two Variables

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Undergraduate Math Seminar - University of Pittsburgh

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(not a proof!)

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Claim: $g(a, b) = ab - a - b$.

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$a(x_0 + bt) + b(y_0 - at) = ax_0 + abt + by_0 - bat = ax_0 + by_0 = n$ by definition of (x_0, y_0).

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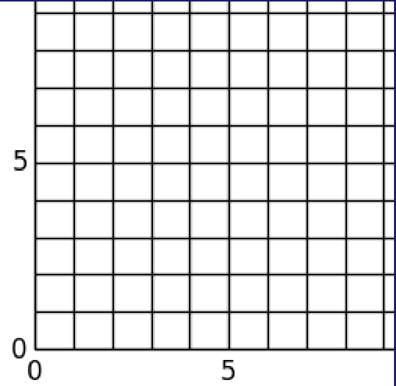
- Solving for y above implies $y = \frac{n - ax}{b} = \frac{n}{b} - \frac{a}{b}x$.
- Focus on the first quadrant (i.e. $x, y \geq 0$) and look what happens when n increases.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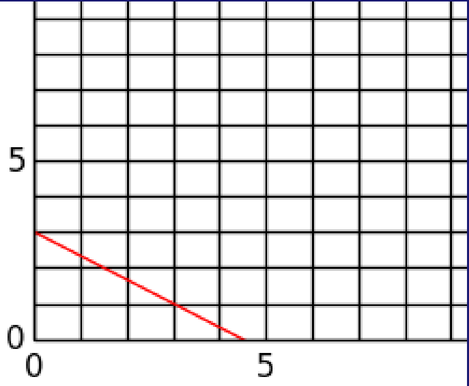
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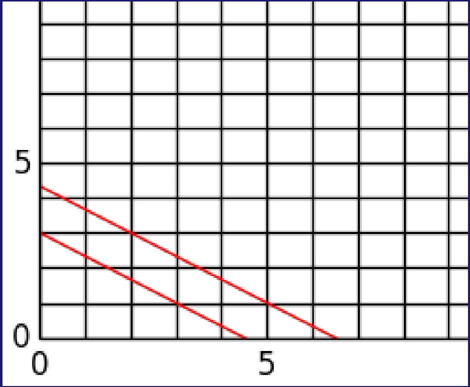
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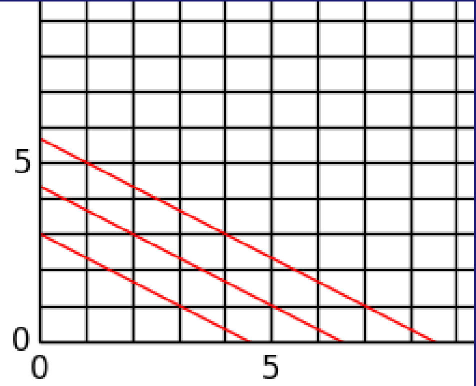
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If you start at some point on the line $ax + by = n$ *and travel a distance* $\sqrt{a^2 + b^2}$*, you *must* pass through an integral solution.*

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Let $S = \{(x, y) : ax + by = n; x, y \geq 0\}$ be the section of the line that lies in the first quadrant.

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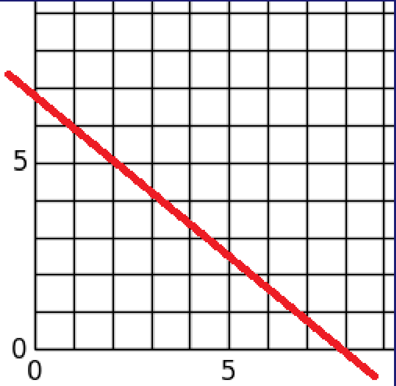
Let's compute the length of S, denoted by $|S|$.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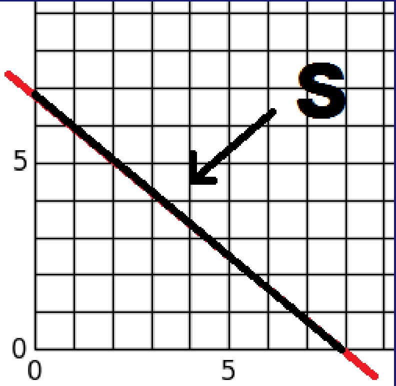
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Hence, for any $n \geq ab, |S| \geq L = \sqrt{a^2 + b^2}$, meaning there must be an integral solution on S.

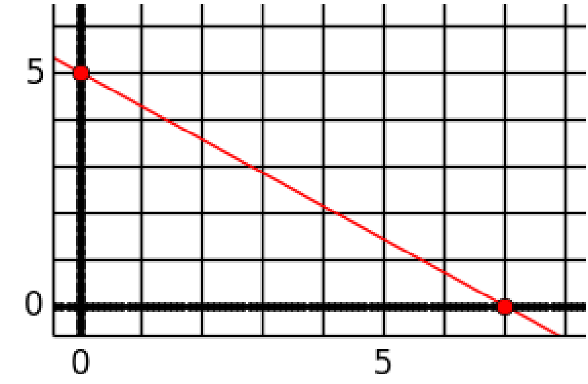


Figure: Example with $a = 5, b = 7$, and $n = 35$.

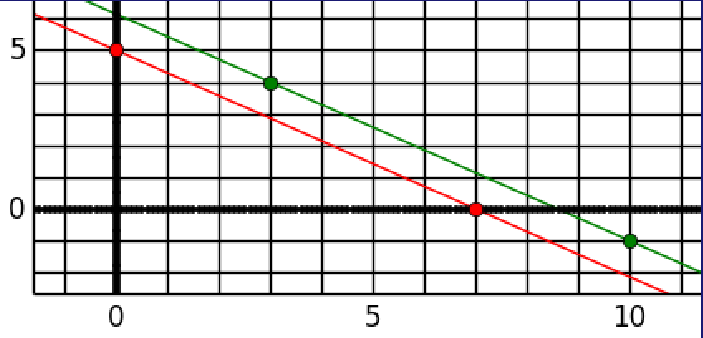


Figure: Example with $a = 5, b = 7$, $n = 35$, and $n = 43$.

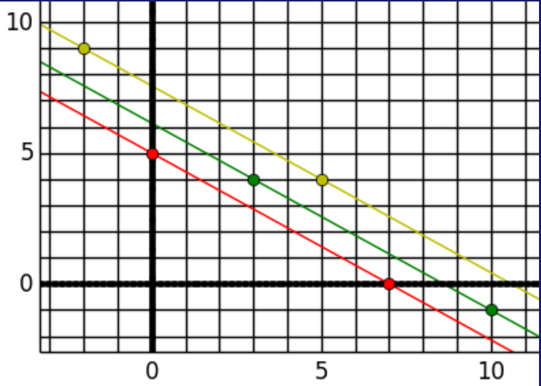
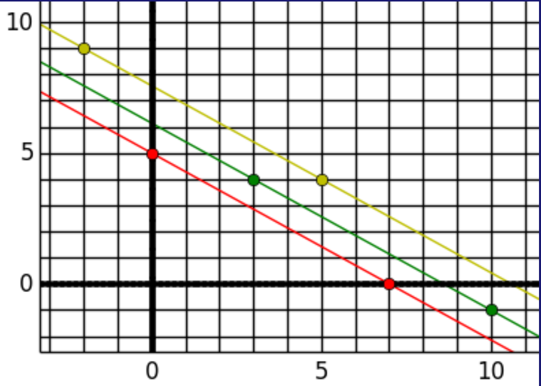


Figure: Example with $a = 5, b = 7$, $n = 35$, $n = 43$, and $n = 54$.



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Thus, $g(a, b) < ab$.

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- $(-1, a-1)$ and $(b-1, -1)$ are integer solutions to this equation.
- Not only that, but they are consecutive! (Remember $(x_0, y_0) \implies (x_0 + b, y_0 - a)$)

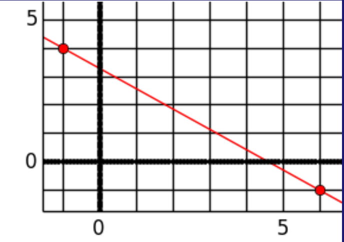
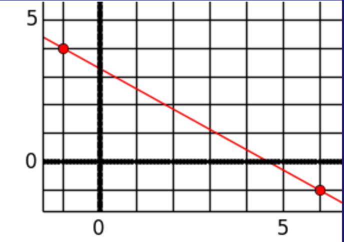


Figure: Example with $a = 5, b = 7$.



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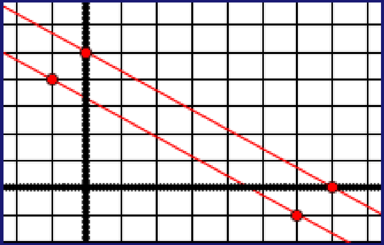
What if $ab - a - b < n < ab$?



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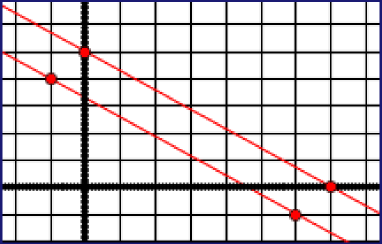

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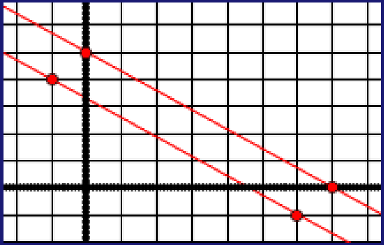
Both of these lines are parallel (the slope is $-\frac{a}{b}$).



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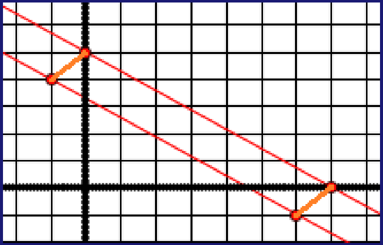
Both of these lines are parallel (the slope is $-\frac{a}{b}$). So the shape connecting all four points is a parallelogram.



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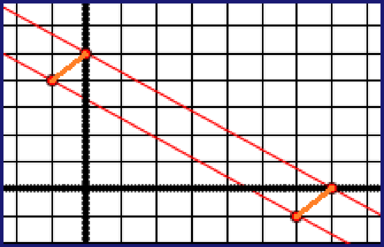

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Since the longer side is $\sqrt{a^2 + b^2}$, for $ab - a - b < n < ab$, there must be a solution inside the parallelogram for each n.

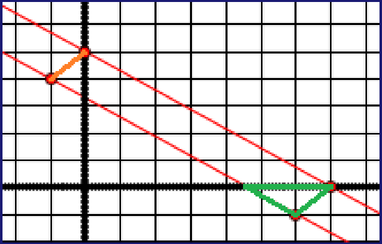
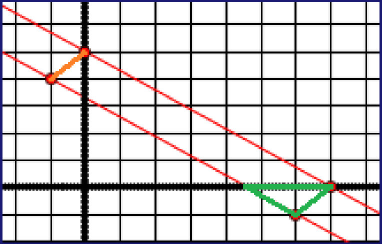


Figure: Example with $a = 5, b = 7$. (lower triangle)



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Lower Triangle: Any solution inside will have $-1 < y < 0$.

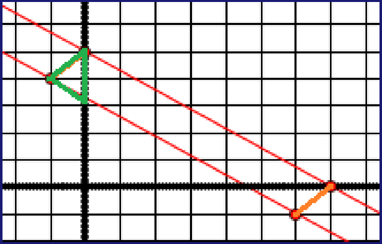
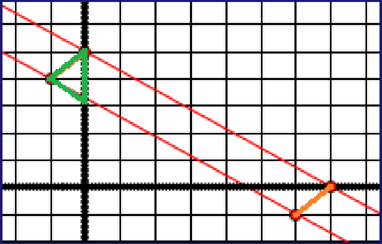


Figure: Example with $a = 5, b = 7$. (upper triangle)



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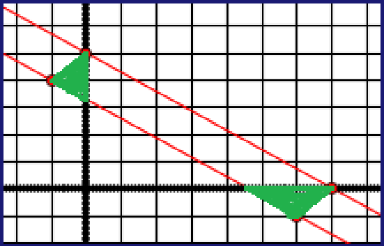
Upper Triangle: Any solution inside will have $-1 < x < 0$.



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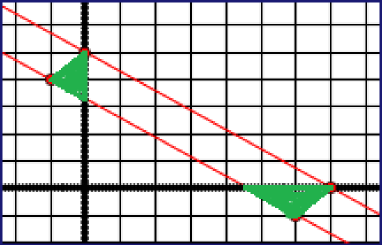

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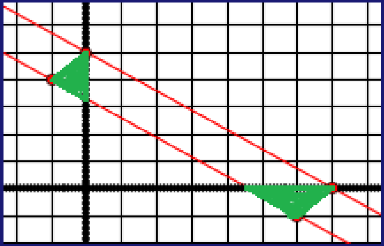

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- Harder than one may think.

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- $g(6, 9, 20) = 43$.

Formula for $g(a, b, c)$?

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- Must be commutative.
- $g(a, b, c) \leq g(a, b)$.

There is **no** known explicit formula for $g(a, b, c)$.

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- Upper bound (Erdős, Graham in *Acta Arithmetica 1972*):
  $g(a_1, ..., a_n) \leq 2a_{n-1} \left\lfloor \frac{a_n}{n} \right\rfloor - a_n$ where $a_1 < a_2 < ... < a_n$.

Extensions:

Arithmetic sequences: Given a, d, s with $\gcd(a, d) = 1$,

$$g(a, a+d, a+2d, \ldots, a+sd) = \left(\left\lfloor \frac{a-2}{s} \right\rfloor + 1 \right) a + ad - a - d.$$

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Geometric sequences: Given m, n, p, with $\gcd(m, n) = 1$,

$$g(m^p, m^{p-1}n, m^{p-2}n^2, \ldots, mn^{p-1}, n^p)$$
$$= n^{p-1}(mn - m - n) + \frac{m^2(n-1)(m^{p-1} - n^{p-1})}{m - n}$$

Extensions:
There are $(a-1)(b-1)/2 = (g(a,b)+1)/2$ natural numbers that will never be attained. For 3 and 7, $\{1,2,4,5,8,11\}$ are never achieved, a total of $(11+1)/2 = 6$ numbers.

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What is the largest number n that has a *unique* solution (x,y) to $ax + by = n$? (Call it $h(a,b)$)
Did some coding... $h(a,b) = 2ab - a - b$. For example, with 3 and 7, 32 is the largest number with a unique pairing:

$32 = 3*6 + 7*2$. All larger numbers have more than one solution.
$33 = 3*11 + 7*0 = 3*4 + 7*3$. This can be good for making bets...

Thank You!