THE 1877 BOUSSINESQ CONJECTURE: TURBULENT FLUCTUATIONS ARE DISSIPATIVE ON THE MEAN FLOW

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Abstract. The 1877 assumption/conjecture of Boussinesq is that turbulent fluctuations have a dissipative effect on the mean flow. The eddy viscosity hypothesis is that this dissipative effect can be modelled by a turbulent viscosity term. Many computational and experimental data sets have shown that the eddy viscosity hypothesis is not generally correct. The inverse cascade of energy to larger scales in $2d$ flows is thought to imply that *both* are false for $2d$ flows. This article gives a proof that for strong solutions in $2d$ and $3d$ the Boussinesq assumption does holds in a time averaged sense. Since this proof contradicts commonly held interpretations of the Batchelor-Leith-Kraichnen theory of $2d$ turbulence, this (extended) report includes a brief discussion of BLK theory in the light of the theorem.

Key words. Boussinesq assumption, eddy viscosity hypothesis, anomalous dissipation

1. Introduction. The Boussinesq assumption, developing ideas of Saint-Venant [17], is often expressed as "*turbulent fluctuations are dissipative on the mean flow*". In this report we give a proof of this 1877 conjecture in both 2d and 3d for ensemble averages. (The formal steps of the proof also hold for Reynolds / long time averages with the issue of existence of the limits as $T \to \infty$.) The Boussinesq assumption / eddy viscosity hypothesis has been viewed as an engineering approximation that, while false, was good enough when enough free parameters are data-fit to a specific flow. That the Boussinesq assumption holds in 2d is surprising since it contradicts the common interpretation of the Batchelor-Leith-Kraichnan (BLK) phenomenology of 2d turbulence, see Section 4.

The eddy viscosity hypothesis is that this dissipativity can be modelled by a viscosity term with particular flow features captured by a turbulent viscosity coefficient. These assumptions are the foundation most turbulence models used in practical computations. Aside from simple eddy viscosity, turbulence models increasingly use a variety of mechanisms for dissipativity (tensor diffusion, hyper viscosity, variational multi-scale diffusion, time relaxation, terms analogous to ones arising in viscoelastic flows, e.g., [10], [9], [16], [8], [1], [19][14], [15]). Thus, it is worthwhile to consider the Boussinesq assumption (proven herein) separately from the eddy viscosity hypothesis.

Boussinesq [2] justified dissipativity (and a mixing length eddy viscosity model) by an analogy with the kinetic theory of gasses where the micro-scale of turbulence is the mixing length. If there is a scale separation, mathematical support exists for both, well summarized in Chapters 11 and 12 in [14], see also [4]. Boussinesq's assumption of scale separation has been criticized since turbulent flows contain a near continuum of scales, e.g., [12], [13], [18]. Validity of the eddy viscosity hypothesis has been checked against experimental data and numerically generated flow data in various places and also found not to hold generally, e.g., [13], [18]. Furthermore, in $2d$ turbulence the conventional phenomenology is that (in the absence of boundaries) energy cascades from small scales to larger scales. For $2d$ turbulent flows negative viscosity has been even suggested, e.g., [20]. We quote [18] (about the eddy viscosity hypothesis):

> " ... *The results using numerical ... or experimental data are very consistent in pointing the non-validity of the Boussinesq hypothesis...* "

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Given this work, it is perhaps surprising that we give here a proof of the Boussinesq assumption in *both* $2d$ and $3d$.

1.1. Preliminaries. Let Ω denote a $2d$ or $3d$ flow domain, ν the kinematic viscosity, $f(x)$ the body force, $u_j(x,t), (j=1,\cdots,J)$, a collection of velocities. Let the ensemble of initial conditions determining the ensemble of velocities by

$$u_j(x,0) = u_j^0(x), \quad j = 1, \cdots, J, \text{ in } \Omega. \tag{1.1}$$

We assume[1] $f \in L^2(\Omega)$ and let $||\cdot||$ denote the usual L^2 norm. The velocities and pressures u_j, p_j satisfy the Navier-Stokes equations under no slip (or periodic) boundary conditions $u_j = 0$ on $\partial\Omega$ and

$$\begin{aligned} u_{j,t} + u_j \cdot \nabla u_j - \nu \triangle u_j + \nabla p_j &= f(x), \\ \nabla \cdot u_j &= 0 \text{ in } \Omega \\ u_j &= 0 \text{ on } \partial\Omega\ . \end{aligned} \tag{1.2}$$

Definition 1.1 (Means and Fluctuations). *Ensemble averages* $\langle v_j \rangle$ *and associated fluctuations* v_j' *are*

$$\langle v_j \rangle := \frac{1}{J} \sum_{j=1}^{J} v_j \quad \textit{and} \quad v_j' = v_j - \langle v \rangle\, .$$

Since $\langle v_j \rangle$ is independent of j we shall often write $\langle v \rangle$. Ensemble averaging satisfies well known properties, [3], [14],

$$\begin{array}{cc} \langle \langle v \rangle \rangle = \langle v \rangle & \\ \langle v_j' \rangle = 0 & \\ \langle \langle w \rangle \otimes v_j \rangle = \langle w \rangle \otimes \langle v \rangle & \langle \langle w \rangle \cdot v_j \rangle = \langle w \rangle \cdot \langle v \rangle \\ \langle \langle w \rangle \otimes v_j' \rangle = \langle w \rangle \otimes \langle v_j' \rangle = 0 & \langle \langle w \rangle \cdot v_j' \rangle = \langle w \rangle \cdot \langle v_j' \rangle = 0 \\ \frac{\partial}{\partial x} \langle v \rangle = \left\langle \frac{\partial}{\partial x} v \right\rangle & \frac{\partial}{\partial t} \langle v \rangle = \left\langle \frac{\partial}{\partial t} v \right\rangle \end{array}$$

The effect of the fluctuations on the mean flow are connected to the Reynolds stresses.

Definition 1.2 (Reynolds stresses). *The Reynolds stresses are*

$$R(u,u) := \langle u_j \rangle \otimes \langle u_j \rangle - \langle u_j \otimes u_j \rangle\, .$$

Using the above properties of averaging, it is not hard to show (by using the above properties of averaging, expanding $u_j = \langle u_j \rangle + u_j'$ and cancelling terms) that

$$R(u,u) = -\left\langle u_j' \otimes u_j' \right\rangle .$$

This does not hold for averaging by a convolution (local spacial or local time filter) commonly used in large eddy simulation. One important quantity that the proof identifies is the Variance of the velocity and velocity gradient.

[1] The result also holds if $f = f(x,t) \in L^\infty(0,\infty; L^2(\Omega))$.

Definition 1.3 (Variance). *The Variance of u and ∇u are respectively*

$$Var(u) := \left\langle \int_\Omega |u_j|^2 dx \right\rangle - \int_\Omega |\langle u_j \rangle|^2 dx$$
$$Var(\nabla u) := \left\langle \int_\Omega |\nabla u_j|^2 dx \right\rangle - \int_\Omega |\nabla \langle u \rangle|^2 dx$$

The variance satisfies

$$Var(u) \geq 0 \text{ and } Var(\nabla u) \geq 0.$$

Further, using the fact that the L^2 norm comes from an inner product, expanding $u_j = \langle u_j \rangle + u_j'$, calculating

$$Var(u) = \left\langle \int_\Omega (\langle u_j \rangle + u_j') \cdot (\langle u_j \rangle + u_j') dx \right\rangle - \int_\Omega \langle u_j \rangle \cdot \langle u_j \rangle \, dx$$

and cancelling terms we easily find the standard result that the variance measures fluctuations

$$Var(u) = \left\langle \int_\Omega |u_j'|^2 dx \right\rangle \text{ and } Var(\nabla u) = \left\langle \int_\Omega |\nabla u_j'|^2 dx \right\rangle.$$

1.2. Mathematical Formulation. Ensemble averaging the NSE gives

$$\langle u \rangle_t + \langle u \rangle \cdot \nabla \langle u \rangle - \nu \triangle \langle u \rangle - \nabla \cdot R(u,u) + \nabla \langle p \rangle = f(x) \text{ and } \nabla \cdot \langle u \rangle = 0 \text{ in } \Omega.$$

Taking the dot product with $\langle u \rangle$, integrating over the flow domain Ω and integrating by parts gives the energy equation for the mean flow

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\langle u \rangle|^2 dx + \int_\Omega \nu |\nabla \langle u \rangle|^2 dx + \int_\Omega R(u,u) : \nabla \langle u \rangle \, dx = \int_\Omega f(x) \cdot \langle u \rangle \, dx. \quad (1.3)$$

The first term, $\frac{d}{dt}\frac{1}{2}\int |\langle u \rangle|^2 dx$, is the rate of change of kinetic energy of the mean flow. The second, $\int \nu |\nabla \langle u \rangle|^2 dx$, is the rate of energy dissipation (since the term is non-negative) of the mean flow due to molecular viscosity and the last term, $\int f(x) \cdot \langle u \rangle \, dx$, is the rate of energy input:

$$\frac{d}{dt}\frac{1}{2}\int |\langle u \rangle|^2 dx : \text{ rate of change of kinetic energy of the mean flow,}$$
$$\int \nu |\nabla \langle u \rangle|^2 dx : \text{ rate of energy dissipation by viscosity,}$$
$$\int f(x) \cdot \langle u \rangle \, dx : \text{ rate of energy input,}$$
$$\int_\Omega R(u,u) : \nabla \langle u \rangle \, dx : \text{ effect of fluctuations upon the mean flow's energy.}$$

The third term, $\int R(u,u) : \nabla \langle u \rangle \, dx$, is the effect of fluctuations upon the mean flow. LIM denotes a Banach or generalized limit and will be used for long time averages, e.g., [7], [6]. We prove the conjecture that this term dissipates energy in the mean flow in a time averaged sense

$$LIM_{T \to \infty} \frac{1}{T} \int_0^T \int_\Omega R(u,u) : \nabla \langle u \rangle \, dx dt \geq 0.$$

In particular, this implies

$$\lim_{T\to\infty}\inf \frac{1}{T}\int_0^T\int_\Omega R(u,u):\nabla\langle u\rangle\, dxdt \geq 0.$$

Key Assumptions. The two key assumptions are:

A1. *The ensemble is generated by different initial data and not different body forces or physical parameters.*

A2. *The solutions of the NSE are strong solutions or sufficiently regular that the energy equality holds.*

For A2, we note that the gap between the energy equality and the energy inequality is one of the (many) mysteries of the Navier Stokes equations.

THEOREM 1.4. *Under assumptions **A1, A2***

$$LIM_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega R(u,u):\nabla\langle u\rangle\, dxdt =$$
$$= LIM_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega \left\langle \nu|\nabla u_j'|^2\right\rangle dxdt \geq 0.$$

There is an obvious extension to inclusion of a random variable in the initial condition. The result could be phrased (as noted above) as the limit infimum being non-negative. This proves that in both $2d$ and $3d$ on bounded domains the **time averaged effects of the Reynolds stresses is to dissipate energy**. In other words, *turbulent fluctuations are dissipative in the mean on the mean flow.* At any instant in time or at any specific point in space, the effect of the turbulent fluctuations may be positive or negative. The theorem does not address the correct parametrization of that dissipation. As noted above, many dissipative mechanisms have been studied to parameterize their effects.

2. Consequences of the proof. Variance Evolution. The proof is very short and will be in the next section. It has interesting consequences. In particular, it shows that the variance obeys an important evolution equation

$$\textbf{Variance Evolution}:$$
$$\frac{1}{2}Var(u(T))+\nu\int_0^T Var(\nabla u(t))dt = \frac{1}{2}Var(u(0))+\int_0^T\int_\Omega R(u,u):\nabla\langle u\rangle\, dxdt.$$

In its most compact form, the proof consists of establishing this relation, dividing by T then taking the limit as $T\to\infty$ (i.e.., time averaging this relation). The one time-step version of this equation is

$$\frac{1}{2}Var(u(t^{n+1}))-\frac{1}{2}Var(u(t^n))+\nu\int_{t^n}^{t^{n+1}} Var(\nabla u(t))dt =$$
$$=\int_{t^n}^{t^{n+1}}\int_\Omega R(u,u):\nabla\langle u\rangle\, dxdt.$$

It is possibly important to notice that the LHS is computable from information available at reach time-step within a calculation. Its magnitude can be used to scale the closure model used for the Reynolds stresses (on the RHS) and its sign to indicate whether there is forward scatter or backscatter at that instant/time-step).

Reynolds / long time averages. Following the formal steps of the proof, when the averaging operator is the classical Reynolds averaging, given by

$$\langle v\rangle (x) := \lim_{T\to\infty} \frac{1}{T}\int_0^T v(x,t)dt,$$

the same result holds. This result is formal in that it assumes that at each step the long time limits are well defined.

Turbulence intensities. Recall that the turbulence intensity is defined by

$$I(u(t)) := \frac{\left\langle ||u_j'(t)||^2\right\rangle}{||\langle u\rangle||^2}$$

The proof suggests that $I(\nabla u)$ is a better measure of turbulence intensity than $I(u)$.

3. The proof. We calculate the energy equality for $\left\langle \frac{1}{2}||u_j||^2\right\rangle$ and for $\frac{1}{2}||\langle u_j\rangle||^2$. By subtracting, we obtain an estimate for the variance of ∇u_j which will then be time averaged.

Take the dot product of the NSE with u_j and integrate over Ω then ensemble average the result and average in time. Standard manipulations yield

$$\frac{1}{T}\left\{\left\langle \frac{1}{2}||u_j(T)||^2\right\rangle - \left\langle \frac{1}{2}||u_j^0||^2\right\rangle\right\} + \frac{1}{T}\int_0^T \left\langle \nu||\nabla u_j(t)||^2\right\rangle dt = \tag{3.1}$$
$$= \frac{1}{T}\int_0^T\int_\Omega f(x)\cdot\langle u\rangle\, dxdt.$$

Next, integrate (1.3) over $[0,T]$ and divide by T. This gives

$$\frac{1}{T}\left\{\frac{1}{2}||\langle u\rangle (T)||^2 - \frac{1}{2}||\left\langle u_j^0\right\rangle||^2\right\} + \frac{1}{T}\int_0^T \nu||\nabla\langle u\rangle (t)||^2 dt + \tag{3.2}$$
$$+\frac{1}{T}\int_0^T\int_\Omega R(u,u):\nabla\langle u\rangle\, dxdt = \frac{1}{T}\int_0^T\int_\Omega f(x)\cdot\langle u\rangle\, dxdt.$$

Consider the terms in the above. Since $||\langle u\rangle|| \leq \langle ||u_j||\rangle \leq C$ the first term (in braces) $\to 0$ as $T\to\infty$. For the same reason the second term is uniformly bounded in T as is the last term (on the RHS). Thus the third, Reynolds stress, term is also uniformly bounded in T. Thus, generalized limits of this term exist as $T\to\infty$. Subtracting (3.1) - (3.2) gives

$$\frac{1}{T}\left\{\left\langle \frac{1}{2}||u_j(T)||^2\right\rangle - \left\langle \frac{1}{2}||u_j^0||^2\right\rangle - \frac{1}{2}\int_\Omega |\langle u\rangle (T)|^2 dx + \frac{1}{2}\int_\Omega |\langle u\rangle (0)|^2 dx\right\}$$
$$+\frac{\nu}{T}\int_0^T \left(\left\langle ||\nabla u_j(t)||^2\right\rangle - ||\nabla\langle u\rangle (t)||^2\right) dt =$$
$$= \frac{1}{T}\int_0^T\int_\Omega R(u,u):\nabla\langle u\rangle\, dxdt.$$

As $T\to\infty$ and the first term (in braces) $\to 0$. Thus, we have

$$LIM_{T\to\infty}\frac{\nu}{T}\int_0^T \left(\left\langle ||\nabla u_j(t)||^2\right\rangle - ||\nabla\langle u\rangle (t)||^2\right) dt =$$
$$= LIM_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega R(u,u):\nabla\langle u\rangle\, dxdt.$$

The LHS integrand is the variance of ∇u_j so that $\left\langle ||\nabla u_j(t)||^2\right\rangle - ||\nabla \left\langle u\right\rangle(t)||^2 = \left\langle ||\nabla u_j'(t)||^2\right\rangle$. Thus, as claimed,

$$LIM_{T\to\infty}\frac{1}{T}\int_0^T\int_\Omega R(u,u):\nabla\left\langle u\right\rangle dxdt =$$
$$= LIM_{T\to\infty}\frac{\nu}{T}\int_0^T \left(\left\langle ||\nabla u_j(t)||^2\right\rangle - ||\nabla\left\langle u\right\rangle(t)||^2\right)dt =$$
$$= LIM_{T\to\infty}\frac{\nu}{T}\int_0^T \left\langle ||\nabla u_j'(t)||^2\right\rangle dt \geq 0.$$

4. A few consequences. That fluctuations are dissipative in the time averaged mean on the mean flow has several consequences. First, it suggests that care must be used in applying the conventional phenomenology of $2d$ turbulence (which assumes an unbounded domain and predicts a cascade of energy to larger scales) to flow in bounded domains. That dissipativity holds in a time averaged sense means that there is no contradiction with instants where backscatter occurs or with the observation in p.48 of [8] for $2d$ turbulence that "*... eddy viscosity is frequently negative ...*". The theorem also does not give insight into which scales the dissipation should damp or the form the dissipation should take.

Backscatter. Backscatter describes situations when fluctuations effects on the mean flow increases the energy in the mean flow. The book of Starr [20] collects examples of many flows where backscatter occurs at certain instants and places. The theorem indicates that backscatter is not the averaged behavior in $3d$ or even $2d$. Thus, it makes the study of backscatter more interesting and important.

The Batchelor-Leith-Kraichnan Phenomenology of 2d turbulence. The theorem contradicts the conventional interpretation of the BLK phenomenology which predicts a cascade of energy to larger scales in $2d$. This paragraph presents three speculations on why there is no contradiction with BLK phenomenology but rather with the conventional application of it to the $2d$ NSE in bounded domains. For background, in $3d$ the Kolmogorov phenomenology has been a great success (in spite of the difficulties associated with the fundamental problem of the $3d$ NSE). Whatever $3d$ data is analyzed a $k^{-5/3}$ energy spectrum emerges. In $2d$ the picture has been less clear. In numerics, it is well known that for under-resolved computations in $2d$ (and even $1d$ linear convection diffusion equations), if a clean numerical method is used on an under-resolved mesh, approximations result with artifacts at the smallest resolved scales. These disappear when extra dissipative mechanisms, such as low or even high order upwinding, hyperviscosity and drag terms modelling bottom friction, are included. When the introduced numerical dissipation is greater than that of the effects of the Reynolds stresses, data fitting the resulting flows to determine an effective turbulent viscosity obviously will suggest backscatter. This could be wrongly interpreted as a physical result rather than an indication that numerical anti-diffusion is needed. The second concerns data from soap film experiments[2]. Data from them clearly indicates that the flow of soap films is not governed by the $2d$ NSE. There is significant ($6\% - 10\%$) variation of film thickness and phenomena in soap films, such as hysteresis in vortex shedding, occur that are not contained in the $2d$ NSE.

[2] The author has learned a lot about these from his collegue Professor Walter Goldburg from the Physics departmet. He is grateful for many conversations. Naturally, this paragraphh expresses the author's wild speculations, not those of Professor Goldburg.

Drag occurs with the air on both sides of the film (a 3d effect that must be modelled in a $2d$ simulation). Thus, there are several reasons why soap film data cannot be directly compared with the result of the theorem. Finally, the $2d$ BLK phenomenology addresses $2d$ flow in an infinite domain. The question naturally arises: What happens, in a finite domain, when the cascade to larger scales reaches the domain size? The convention expectation is that the cascade to larger scales continues and there is an accumulation of energy at the largest scales. (This picture is reinforced by the self–organization hypothesis which is addressed next.) The theorem does not apply to the BLK setting but does prove decisively that this interpretation for bounded domains is wrong.

The self-organization hypothesis. One striking example of an apparent contradiction with physical effects is the self organization hypothesis of $2d$, periodic, *unforced*/decaying turbulence. For $2d$ periodic decaying turbulence it has been observed, explained and proven by van Groesen [25] (see also [21], [22], [23], [24] for important related work) that the flow evolves to the largest Stokes vortex allowed by the size of the domain. (A numerical test of Jiang [26] suggests it holds under no slip boundary conditions as well.) This effect is not connected to an energy cascade to larger scales, [25], [21], [22], [23], [24]. Self-organization is a diffusive process. The emergence of the largest Stokes vortex is because $2d$ flow has two integral invariants that decay at different rates in the unforced case. Energy decays slowly while enstrophy decays more rapidly. Thus, energy in the resolved scales does not increase due to either backscatter or an inverse cascade.

Calibration of Closure Models. The second consequence is that the theorem does specify exactly how much turbulent dissipation a model should introduce. If the time averaged amount is applied at each instant (an assumption of statistical equilibrium), the rate of energy dissipation done by the modelling terms that replace the Reynolds stress terms should be exactly

$$\nu \left\langle ||\nabla u_j'(t)||^2 \right\rangle .$$

For example, suppose a turbulent viscosity term with constant ν_T replaces the Reynolds stresses

$$R(u,u) \Leftarrow -\nabla \cdot (\nu_T \nabla \langle u \rangle) \ \ (+ \text{ terms incorporated into the pressure})$$

The dissipation added to the resulting model would be (for constant ν_T)

$$\int_\Omega R(u,u) : \nabla \langle u \rangle \, dx \Leftarrow \nu_T ||\nabla \langle u \rangle ||^2.$$

Recall that the turbulence intensity is defined by

$$I(u(t)) := \frac{\left\langle ||u_j'(t)||^2 \right\rangle}{|| \langle u \rangle ||^2}$$

We calculate, under these simplifications, that the correct turbulent viscosity coefficient must satisfy

$$\nu_T ||\nabla \langle u \rangle ||^2 = \nu \left\langle ||\nabla u_j'(t)||^2 \right\rangle \text{ which implies}$$

$$\nu_T = \nu \frac{\left\langle ||\nabla u_j'(t)||^2 \right\rangle}{||\nabla \langle u \rangle ||^2} = \nu I(\nabla u).$$

This calculation suggests that $I(\nabla u)$ is a better measure of turbulence intensity than $I(u)$. A similar calculation can be used to scale other model's parameters.

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