A Crank-Nicolson Leap-Frog stabilization: unconditional stability and two applications[1] [2]

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Abstract

We propose and analyze a linear stabilization of the Crank-Nicolson Leap-Frog (CNLF) method that removes all timestep / CFL conditions for stability and controls the unstable mode. It also increases the SPD part of the linear system to be solved at each time step. We give a proof of unconditional stability of the method as well as a proof of unconditional, asymptotic stability of both the stable and unstable modes. We illustrate two applications of the method: uncoupling groundwater - surface water flows and Stokes flow plus a Coriolis term.

1. Introduction

The implicit explicit (IMEX) combination of Crank-Nicolson and Leap Frog (CNLF) is widely used in atmosphere, ocean and climate codes, e.g., [1], [2], [3], [4] and has recently been used for uncoupling groundwater-surfacewater flows, [5]. Although first analyzed in 1963 [6], stability of (CNLF) for systems was only recently proven [7], along with asymptotic stability analysis of the unstable $(u^{n+1} - u^{n-1})$ mode in [8]. (CNLF) has two limitations. First, the unstable mode (for which $u^{n+1} + u^{n-1} \equiv 0$) of LF is not damped by CN unless the CFL condition $\Delta t|\text{Wave Speed}| < 1$ is met. Thus, modular time filters, like the Roberts-Asselin-Williams (RAW) filter [1], [2], [4], have been developed. Second, the CFL restriction, $\Delta t|\text{Wave Speed}| < 1$, even including time filters like RAW, can be too restrictive.

[1]This report is in final form.

[2]The research of the authors was partially supported by NSF grant DMS 1216465 and AFOSR grant FA 9550-12-1-0191.

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This report presents a new stabilization of (CNLF) addressing both issues. (CNLFSTAB) is unconditionally (no CFL condition) stable (Theorem 1) and all modes, including the unstable mode, are unconditionally asymptotically stable (Theorem 2). We give the method (CNLFSTAB) below and prove unconditional stability and control of both the stable and unstable modes in Section 2. Then we test its effects for two important examples: uncoupling evolutionary groundwater - surface water flows, and Stokes flow + strong rotation in Section 3.

Consider an evolution equation written as

$$\frac{du}{dt} + Au + \Lambda u = 0. \tag{1}$$

We assume that $X \hookrightarrow L \hookrightarrow X'$ are Hilbert spaces. Let $< \cdot, \cdot >, || \cdot ||$ denote the inner product and norm on L. Suppose the linear operators A, Λ:

$$A : X \to X' \text{ satisfies } \langle A(u), u \rangle \geq 0 \text{ for all } u \in X, \qquad \text{(POSITIVITY)}$$

$$\Lambda : L \to L \text{ is a bounded, skew symmetric operator,} \qquad \text{(SKEWSYMMETRY)}$$

where the X, X' duality pairing is an extension of the L inner product. These two assumptions ensure that

$$||u(t)||^2 \leq ||u_0||^2, ||u(t)||^2 = ||u_0||^2 \text{ if } Au \equiv 0 \text{ and}$$
$$\|u(t)\| \to 0 \text{ as } t \to \infty \text{ if } A \text{ is SPD.}$$

These are the basic stability properties that must be preserved under discretization.

The stabilized (CNLF) method we consider is then, given $u^0, u^1 \in X$ find $u^n \in X$ for $n \geq 2$ satisfying

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} + \mathbf{\Delta t \Lambda^* \Lambda} \left(\mathbf{u}^{n+1} - \mathbf{u}^{n-1}\right) + A\left(\frac{u^{n+1} + u^{n-1}}{2}\right) + \Lambda u^n = 0. \tag{CNLFSTAB}$$

The stabilization occurs in the term (in bold) $\Delta t \Lambda^* \Lambda \left(u^{n+1} - u^{n-1}\right)$. This term is linear and SPD in the unknown u^{n+1}; it has no undetermined tuning parameters; the extra consistency error it contributes is formally $\Delta t^2 \Lambda^* \Lambda (u_t) = O(\Delta t^2)$ which is the same order as (CNLF). (CNLFSTAB), like (CNLF), is a 3 level method and approximations of appropriate accuracy are needed at the first two time steps, e.g. [9]. The stabilization we study herein is a complementary tool similar in spirit to work in [10], [11], [12], [13]. For a general theory of IMEX methods see [9], [14], [15], [16], [17], [18]. Both the algorithm and the stability result in (Theorem 1) extend easily to the case with nonzero right hand side $f(t)$. An extension of Theorems 1 and 2 is given in Remark 1 following Theorem 2 for when $f(t) \to f_\infty$ as $t \to \infty$.

1.1. The usual (CNLF) *method*

The usual (CNLF) method is

$$\frac{u^{n+1}-u^{n-1}}{2\Delta t}+A\left(\frac{u^{n+1}+u^{n-1}}{2}\right)+\Lambda u^n=0. \tag{CNLF}$$

Let $||\Lambda||=\sup_{0\neq v\in L}||\Lambda v||/||v||<\infty$ denote the operator norm of Λ. Stability analysis in [7] shows (CNLF) is energy stable under the timestep restriction

$$\Delta t||\Lambda||<1 \tag{CFL}$$

long expected, e.g., [6], from root condition analysis. In the common case when Λ is a discretization of a wave propagation problem (CFL) reduces to a CFL type condition like $\Delta t|\text{Wave Speed}|<1$.

Energy stability of (CNLF) under (CFL) is not completely descriptive of computational practice with (CNLF) however. It has long been noted that (CNLF) is marginally stable (described in [19] as "slightly unstable"). When the linear term includes some sort of viscous mechanism

$$\langle Au,u\rangle\geq\alpha||u||^2 \text{ for some } \alpha>0 \text{ and all } u\in X,$$

then $||u(t)||\to 0$ as $t\to\infty$. In this common case, (CNLF) damps the energy in the mode $u^{n+1}+u^{n-1}$ and growth in the unstable mode is often reported. When (CFL) holds, it was shown in [8] that (CNLF) does, in fact, control both the stable and unstable modes. Alternately, if (CFL) is even slightly violated, instability is exhibited in the (undamped) unstable mode, $u^{n+1}-u^{n-1}$. This drawback has led to various fixes such as the RAW or Roberts-Asselin-Williams time filter, see [3], [20], [4]. The stabilization herein is not modular (unlike time filters) but it does remove all CFL timestep conditions (also unlike time filters) and thus provides a tool with complementary strengths and weaknesses to time filters.

2. Stability without the CFL condition

We prove unconditional stability of (CNLFstab). The proof shows that the coefficient of the stabilization term (here taken to be 1) may be reduced retaining unconditional stability. It also shows that if $Au\equiv 0$, then the following quantity is exactly conserved:

$$\frac{1}{2}[||u^{n+1}||^2+||u^n||^2+2\Delta t^2\left(||\Lambda u^{n+1}||^2+||\Lambda u^n||^2\right)]+\Delta t\left\langle\Lambda u^n,u^{n+1}\right\rangle.$$

Theorem 1. *Consider (1.1) under (*Positivity*) and (*SkewSymmetry*). The method* (CNLFstab) *is unconditionally stable (with no timestep restriction): for every $N\geq 1$*

$$\begin{aligned}&\tfrac{1}{2}||u^{N+1}||^2+||u^N||^2+2\Delta t^2||\Lambda u^{N+1}||^2\\&\quad\leq\|u^1\|^2+\|u^0\|^2+2\Delta t\left\langle\Lambda u^0,u^1\right\rangle.\end{aligned} \tag{2}$$

Proof. We take the duality pairing of (CNLFstab) with its stable mode, $u^{n+1}+u^{n-1}$ and follow the steps in [7] but with modified treatment of the critical term $\left\langle \Lambda u^n, u^{n+1}+u^{n-1}\right\rangle$. Take the inner product of (CNLFstab) with the stable mode $(u^{n+1}+u^{n-1})$ and multiply through by $2\Delta t$. Add and subtract $||u^n||^2$; this gives

$$\begin{gathered}(||u^{n+1}||^2+||u^n||^2)-(||u^n||^2+||u^{n-1}||^2)+ \\ +2\Delta t^2\left\langle \Lambda^*\Lambda\left(u^{n+1}-u^{n-1}\right), u^{n+1}+u^{n-1}\right\rangle+ \\ +\Delta t\left\langle A(u^{n+1}+u^{n-1}), u^{n+1}+u^{n-1}\right\rangle+2\Delta t\left\langle \Lambda u^n, u^{n+1}+u^{n-1}\right\rangle=0.\end{gathered}$$

The added stability term can be written as

$$\begin{gathered}2\Delta t^2\left\langle \Lambda^*\Lambda\left(u^{n+1}-u^{n-1}\right), u^{n+1}+u^{n-1}\right\rangle \\ =2\Delta t^2\langle\Lambda\left(u^{n+1}-u^{n-1}\right), \Lambda(u^{n+1}+u^{n-1})\rangle \\ =2\Delta t^2\left(||\Lambda u^{n+1}||^2-||\Lambda u^{n-1}||^2\right) \\ =2\Delta t^2\left[\left(||\Lambda u^{n+1}||^2+||\Lambda u^n||^2\right)-\left(||\Lambda u^n||^2+||\Lambda u^{n-1}||^2\right)\right].\end{gathered}$$

Now, define the stabilized system energy

$$E^{n+1/2}:=||u^{n+1}||^2+||u^n||^2+2\Delta t^2\left(||\Lambda u^{n+1}||^2+||\Lambda u^n||^2\right).$$

We thus have

$$\begin{gathered}E^{n+1/2}-E^{n-1/2}+\Delta t\left\langle A(u^{n+1}+u^{n-1}), u^{n+1}+u^{n-1}\right\rangle \\ +2\Delta t\left\langle \Lambda u^n, u^{n+1}+u^{n-1}\right\rangle=0.\end{gathered}$$

Let $C^{n+1/2}:=\left\langle \Lambda u^n, u^{n+1}\right\rangle$; using skew symmetry of Λ we have

$$\left\langle \Lambda u^n, u^{n+1}+u^{n-1}\right\rangle=C^{n+1/2}-C^{n-1/2}.$$

Thus, the stability equation becomes

$$\begin{gathered}E^{n+1/2}-E^{n-1/2}+\Delta t\left\langle A(u^{n+1}+u^{n-1}), u^{n+1}+u^{n-1}\right\rangle \\ +2\Delta t\left(C^{n+1/2}-C^{n-1/2}\right)=0.\end{gathered}$$

Sum the above from $n=1,\ldots,N$ to obtain

$$E^{N+1/2}+2\Delta tC^{N+1/2}+\Delta t\sum_{n=1}^{N}\left\langle A(u^{n+1}+u^{n-1}), u^{n+1}+u^{n-1}\right\rangle=E^{1/2}+2\Delta tC^{1/2}.$$

We show that $E^{N+1/2}+2\Delta tC^{N+1/2}\geq 0$. By repeated application of the Cauchy-Schwarz and Young inequality we have

$$2\Delta tC^{N+1/2}\leq 2\Delta t^2||\Lambda u^N||^2+\frac{1}{2}||u^{N+1}||^2.$$

This implies

$$E^{N+1/2} + 2\Delta t C^{N+1/2} \geq \tfrac{1}{2}||u^{N+1}||^2 + ||u^N||^2 + 2\Delta t^2 ||\Lambda u^{N+1}||^2 \geq 0.$$

Therefore, we have

$$\begin{aligned} 0 \leq \tfrac{1}{2}||u^{N+1}||^2 + ||u^N||^2 + 2\Delta t^2 ||\Lambda u^{N+1}||^2 \\ + \Delta t \sum_{n=1}^{N} \left\langle A(u^{n+1} + u^{n-1}), u^{n+1} + u^{n-1} \right\rangle \leq E^{1/2} + 2\Delta t C^{1/2}, \end{aligned} \tag{3}$$

which implies (2), by the (Positivity) of the operator A. ■

Next we prove unconditional asymptotic stability.

Theorem 2. *Consider* (CNLFstab). *If the operator $A(\cdot)$ is a symmetric positive definite (SPD) linear operator, then both the stable mode and the unstable mode are unconditionally, asymptotically stable:*

$$u^{n+1} + u^{n-1} \longrightarrow 0 \quad \textit{and} \quad u^{n+1} - u^{n-1} \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty, \tag{4}$$

and thus $u^n \to 0$ as $n \to \infty$.

Proof. Take the inner product of (CNLFstab) with the unstable mode $(u^{n+1} - u^{n-1})$ and multiply through by $2\delta\Delta t$ with $\delta > 0$:

$$\begin{aligned} \delta\|u^{n+1} - u^{n-1}\|^2 + 2\delta\Delta t^2 \left\langle \Lambda^*\Lambda(u^{n+1} - u^{n-1}), u^{n+1} - u^{n-1} \right\rangle \\ +\delta\Delta t \left\langle A(u^{n+1} + u^{n-1}), u^{n+1} - u^{n-1} \right\rangle + 2\delta\Delta t \left\langle \Lambda u^n, u^{n+1} - u^{n-1} \right\rangle = 0. \end{aligned} \tag{5}$$

Because A is SPD, $\|u\|_A := \langle Au, u\rangle$ is well defined. After adding and subtracting $2\delta\Delta t\|u^n\|_A^2 \geq 0$,

$$\begin{aligned} &\delta\Delta t \left\langle A(u^{n+1} + u^{n-1}, u^{n+1} - u^{n-1}) \right\rangle \\ &\quad = \delta\Delta t \left[(\|u^{n+1}\|_A^2 + \|u^n\|_A^2) - (\|u^n\|_A^2 + \|u^{n-1}\|_A^2) \right]. \end{aligned}$$

Define $\mathcal{A}^{n+1/2} := \|u^{n+1}\|_A^2 + \|u^n\|_A^2 \geq 0$. Sum (5) from $n = 1, \ldots, N$ to obtain

$$\begin{aligned} \delta \sum_{n=1}^{N} \left[\|u^{n+1} - u^{n-1}\|^2 + 2\Delta t^2 \|\Lambda(u^{n+1} - u^{n-1})\|^2 \right] \\ + 2\delta\Delta t \sum_{n=1}^{N} \left\langle \Lambda u^n, u^{n+1} - u^{n-1} \right\rangle + \delta\Delta t \mathcal{A}^{N+1/2} = \delta\Delta t \mathcal{A}^{1/2}. \end{aligned} \tag{6}$$

Adding (3) to (6) yields

$$
\begin{aligned}
0 &< \tfrac{1}{2}||u^{N+1}||^2 + ||u^N||^2 + 2\Delta t^2||\Lambda u^{N+1}||^2 + \delta\Delta t\mathcal{A}^{N+1/2} + F^N \\
&\quad + \sum_{n=1}^{N}\left[\Delta t\|u^{n+1}+u^{n-1}\|_A^2 + \delta\|u^{n+1}+u^{n-1}\|^2 + 2\delta\Delta t^2\|\Lambda(u^{n+1}+u^{n-1})\|^2\right] \\
&\leq E^{1/2} + 2\Delta t C^{1/2} + \delta\Delta t\mathcal{A}^{1/2},
\end{aligned}
\tag{7}
$$

where

$$
F^N = \sum_{n=1}^{N} 2\delta\Delta t\left\langle \Lambda u^n, u^{n+1}-u^{n-1}\right\rangle.
$$

For F^N, applying Young's inequality gives, for any ϵ, $0<\epsilon<1$

$$
|F^N| \leq \delta\epsilon\sum_{n=1}^{N}\|u^{n+1}-u^{n-1}\|^2 + \frac{\delta}{\epsilon}\sum_{n=1}^{N}\Delta t^2\|\Lambda u^n\|^2. \tag{8}
$$

The second term on the RHS can be rewritten as

$$
\begin{aligned}
\|\Lambda u^n\|^2 &= \|\Lambda(\frac{u^n+u^{n-2}}{2}) + \Lambda(\frac{u^n-u^{n-2}}{2})\|^2 \\
&= 2\|\Lambda(\frac{u^n+u^{n-2}}{2})\|^2 + 2\|\Lambda(\frac{u^n-u^{n-2}}{2})\|^2 - \|\Lambda(u^{n-2})\|^2.
\end{aligned}
$$

Then the bound on F_N becomes

$$
\begin{aligned}
|F^N| \leq \delta\epsilon\sum_{n=1}^{N}\|u^{n+1}-u^{n-1}\|^2 &+ \frac{\delta}{\epsilon}\Delta t^2\|\Lambda u^1\|^2 - \frac{\delta}{\epsilon}\sum_{n=2}^{N}\Delta t^2\|\Lambda(u^{n-2})\|^2 \\
&+ \frac{\delta}{2\epsilon}\sum_{n=2}^{N}\Delta t^2\left(\|\Lambda(u^n+u^{n-2})\|^2 + \|\Lambda(u^n-u^{n-2})\|^2\right).
\end{aligned}
$$

After shifting the index of the second sum, we obtain

$$
\begin{aligned}
\|F^N\| \leq \delta\epsilon\sum_{n=1}^{N}\|u^{n+1}-u^{n-1}\|^2 &+ \frac{\delta}{\epsilon}\Delta t^2\|\Lambda u^1\|^2 - \frac{\delta}{\epsilon}\sum_{n=2}^{N}\Delta t^2\|\Lambda(u^{n-2})\|^2 \\
&+ \frac{\delta}{2\epsilon}\sum_{n=1}^{N-1}\Delta t^2\left(\|\Lambda(u^{n+1}+u^{n-1})\|^2 + \|\Lambda(u^{n+1}-u^{n-1})\|^2\right).
\end{aligned}
\tag{9}
$$

Applying (9) to (7) gives

$$\begin{aligned}
0 &< \tfrac{1}{2}\|u^{N+1}\|^2 + \|u^N\|^2 + 2\Delta t^2\|\Lambda u^{N+1}\|^2 + \delta\Delta t\mathcal{A}^{N+1/2} \\
&+ \sum_{n=1}^{N}\left[\Delta t\|u^{n+1}+u^{n-1}\|_A^2 - \tfrac{\delta\Delta t^2}{2\epsilon}\|\Lambda(u^{n+1}+u^{n-1})\|^2\right] \\
&\sum_{n=1}^{N}\delta\left[(1-\epsilon)\|u^{n+1}-u^{n-1}\|^2 + \Delta t^2\left(2-\tfrac{1}{2\epsilon}\right)\|\Lambda(u^{n+1}-u^{n-1})\|^2\right] \\
&+ \tfrac{\delta\Delta t^2}{2\epsilon}\|\Lambda(u^{N+1}+u^{N-1})\|^2 + \tfrac{\delta\Delta t^2}{2\epsilon}\|\Lambda(u^{N+1}-u^{N-1})\|^2 \\
&\leq E^{1/2} + 2\Delta t C^{1/2} + \delta\Delta t\mathcal{A}^{1/2} + \tfrac{\delta\Delta t^2}{\epsilon}\|\Lambda u^1\|^2.
\end{aligned} \tag{10}$$

Let $\lambda_{min}(A)$ represent the smallest eigenvalue of A. Because A is SPD, $\lambda_{min}(A) > 0$ and so $\|u^{n+1}+u^{n-1}\|_A^2 \geq \lambda_{min}(A)\|u^{n+1}+u^{n-1}\|^2$.
Choose $\epsilon = \frac{1}{4}$ and $\delta = \frac{\lambda_{min}(A)}{4\Delta t\|\Lambda\|^2}$. By Theorem 1, $\frac{1}{2}\|u^{N+1}\|^2+\|u^N\|^2+2\Delta t^2\|\Lambda u^{N+1}\|^2 > 0$ and can thus be dropped on the LHS. Then (10) becomes

$$\begin{aligned}
0 &< \sum_{n=1}^{N}\left[\tfrac{\Delta t\lambda_{min}(A)}{2}\|u^{n+1}+u^{n-1}\|^2 + \tfrac{3\lambda_{min}(A)}{16\Delta t\|\Lambda\|^2}\|u^{n+1}-u^{n-1}\|^2\right] \\
&\leq E^{1/2} + 2\Delta t C^{1/2} + \tfrac{\lambda_{min}(A)}{4\|\Lambda\|^2}\left(\mathcal{A}^{1/2} + 4\Delta t\|\Lambda u^1\|^2\right).
\end{aligned} \tag{11}$$

The above reduces to

$$\sum_{n=1}^{N}\left[\|u^{n+1}+u^{n-1}\|^2 + \|u^{n+1}-u^{n-1}\|^2\right] \leq C(u^1,u^0), \tag{12}$$

where $C(u^1,u^0)$ is a constant depending on u^1 and u^0 but independent of N. Letting $N\to\infty$, we conclude both $\|u^{n+1}+u^{n-1}\|^2 \to 0$ and $\|u^{n+1}-u^{n-1}\|^2 \to 0$. ■

Remark 1. *The previous conclusions imply asymptotic stability about zero. By linearity, these results extend to nonzero forcing terms on the right hand side,* $f^n = f(t^n)$, *provided* $\sum_{n=1}^{\infty}\|f^n - f_\infty\|_*^2 \leq \mathcal{C}$. *If this holds, then following the steps of Theorems 1 and 2, we conclude that,* $u^{n+1}+u^{n-1}\to 2u_\infty$, $u^{n-1}-u^{n+1}\to 0$, *and* $u^n \to u_\infty$, *where* u_∞ *solves the equilibrium problem,* $Au_\infty + \Lambda u_\infty = f_\infty$.

3. Two Applications

We consider the application of the (CNLFstab) to uncoupling of groundwater-surfacewater flows and to Stokes flow plus a Coriolis force term, an over simplification of the equations of geophysical flow, [21]. The application to Stokes flow + Coriolis force is direct, whereas the Stokes-Darcy application is more technical and the correct extension of the (CNLFstab) method is not obvious. We give a stability analysis of an interpretation of (CNLFstab) for both, incorporating the time and space discretizations.

3.1. The evolutionary Stokes-Darcy problem

See [22], [23], [24], [25], [26], [27], [28] for a careful presentation of the Stokes-Darcy model, the derivation of its variational formulation (which involves a number of technical steps) and its associated numerical analysis. Let Ω_f, Ω_p lie across an interface I from each other. For specificity, we take $\Omega_f = (0,1)\times(0,1)$, $\Omega_p = (0,1)\times(-1,0)$ and $I = \{(x,0), 0 < x < 1\}$. The fluid velocity u, fluid pressure, p, and porous media's piezometric head ϕ satisfy

$$\begin{aligned} u_t - \nu\Delta u + \nabla p = f_f(x,t), \nabla\cdot u = 0, \ \text{ in } \ \Omega_f, \\ S_0\phi_t - \nabla\cdot(\mathcal{K}\nabla\phi) = f_p(x,t), \ \text{ in } \ \Omega_p, \\ \phi(x,0) = \phi_0(x), \ \text{ in } \Omega_p \text{ and } u(x,0) = u_0(x), \ \text{ in } \Omega_f, \\ \phi(x,t) = 0, \ \text{ in } \partial\Omega_p\backslash I \text{ and } u(x,t) = 0, \ \text{ in } \partial\Omega_f\backslash I. \end{aligned} \tag{13}$$

Let $\widehat{n}_{f/p}$ denote the outward unit normal vector on I with respect to each subdomain. The coupling conditions across I are conservation of mass, balance of forces on I and the Beavers-Joseph-Saffman condition on the tangential velocity:

$$\begin{aligned} u\cdot\widehat{n}_f - \mathcal{K}\nabla\phi\cdot\widehat{n}_p = 0 \text{ and } p - \nu\ \widehat{n}_f\cdot\nabla u\cdot\widehat{n}_f = g\phi \text{ on } I, \\ -\nu\ \nabla u\cdot\widehat{n}_f = \frac{\alpha}{\sqrt{\widehat{\tau}_i\cdot\mathcal{K}\cdot\widehat{\tau}_i}} u\cdot\widehat{\tau}_i, \ \text{ on } I \text{ for any } \widehat{\tau}_i \text{ tangent vector on } I. \end{aligned}$$

see [29], [30], [31]. Here g, $\mathcal{K}$, ν and S_0 are the gravitational acceleration constant, hydraulic conductivity tensor, kinematic viscosity and specific mass storativity coefficient, all positive. Often $\lambda_{\min}(\mathcal{K})$ and S_0 are small, [32].

We denote the $L^2(I)$ norm by $||\cdot||_I$ and the $L^2(\Omega_{f/p})$ norms by $||\cdot||_{f/p}$, respectively; the corresponding inner products are denoted by $(\cdot,\cdot)_{f/p}$. To discretize the Stokes-Darcy problem in space by the finite element method we choose conforming velocity, pressure, and Darcy pressure finite element spaces

$$\begin{aligned} \text{Velocity} \ &: \quad X_f^h \subset \ X_f := \{v \in \left(H^1(\Omega_f)\right)^d : v = 0 \text{ on } \ \partial\Omega_f\backslash I\}, \\ \text{Darcy pressure} \ &: \quad X_p^h \subset \ X_p := \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \ \partial\Omega_p\backslash I\}, \\ \text{Stokes pressure} \ &: \quad Q_f^h \subset Q_f \ := L^2(\Omega_f). \end{aligned}$$

X_p^h, X_f^h are separate FEM spaces; continuity across the interface I is not assumed. The Stokes velocity-pressure finite element spaces (X_f^h, Q_f^h) are assumed to satisfy the usual discrete inf-sup condition for stability of the discrete pressure, e.g., [33], [34], [35]. Define

$$\begin{aligned} a_f(u,v) &= (\nu\nabla u, \nabla v)_f + \sum_i \int_I \frac{\alpha}{\sqrt{\widehat{\tau}_i\cdot\mathcal{K}\cdot\widehat{\tau}_i}}(u\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)ds, \\ a_p(\phi,\psi) &= g(\mathcal{K}\nabla\phi, \nabla\psi)_p, \quad \text{and} \\ c_I(u,\phi) &= g\int_I \phi u\cdot\widehat{n}_f ds. \end{aligned}$$

Adaptation to Stokes-Darcy problem: Find $(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ satisfying, for all $v_h \in X_f^h$, $q_h \in Q_f^h$, $\psi_h \in X_p^h$

$$
\begin{aligned}
gS_0\left(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\Delta t}, \psi_h\right)_p + +\Delta t g^2(\nabla(\phi_h^{n+1}-\phi_h^{n-1}), \nabla\psi_h)_p \\
+\Delta t g^2(\phi_h^{n+1}-\phi_h^{n-1}, \psi_h)_p \\
+a_p\left(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}, \psi_h\right) - c_I(u_h^n, \psi_h) = g(f_p^n, \psi_h)_p \qquad \text{(SDSTAB)} \\
\left(\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}, v_h\right)_f + \left(\nabla\cdot\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}, \nabla\cdot v_h\right)_f + a_f\left(\frac{u_h^{n+1}+u_h^{n-1}}{2}, v_h\right) \\
-(\frac{p_h^{n+1}+p_h^{n-1}}{2}, \nabla\cdot v_h)_f + c_I(v_h, \phi_h^n) = (f_f^n, v_h)_f, \\
(q_h, \nabla\cdot u_h^{n+1})_f = 0.
\end{aligned}
$$

The stabilization terms in (SDSTAB) are of the type studied in [10] in the porous medium and grad-div stabilization, [36], of u_t in the fluid region:

$$
\Delta t g^2(\phi_h^{n+1}-\phi_h^{n-1}, \psi_h)_{H^1(\Omega_p)} \quad \text{and} \quad \left(\nabla\cdot\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}, \nabla\cdot v_h\right)_f.
$$

We prove long time, asymptotic stability (over $0 \le t < \infty$) without any time step conditions. Let the $H_{DIV}(\Omega_f)$ norm be denoted

$$
||u||_{DIV} := \sqrt{||u||_f^2 + ||\nabla\cdot u||_f^2}.
$$

The following trace inequality from Moraiti [37], which holds for our Ω_f, Ω_p with constant 1, is essential:

$$
\left|\int_I \phi u\cdot\widehat{n}ds\right| \le ||u||_{DIV}||\phi||_{H^1(\Omega_p)}, \text{ for all } u\in X_f, \phi\in X_p. \qquad \text{(TRACE)}
$$

Remark 2 (On the form of the stabilization). *To implement exactly the stabilization term* $\mathbf{\Delta t \Lambda^* \Lambda}\left(\mathbf{u}^{n+1}-\mathbf{u}^{n-1}\right)$ *for the Stokes-Darcy problem, one must define a linear operator* $\Lambda = (\Lambda_f, \Lambda_p) : X_f^h \times X_p^h \to X_f^h \times X_p^h$ *via the Riesz representation theorem by*

$$
(\Lambda_f(u,\phi), v)_f + (\Lambda_p(u,\phi), \psi)_p = \int_I \psi u\cdot\widehat{n}ds - \int_I \phi v\cdot\widehat{n}ds.
$$

It is not clear even if so defined the result would yield a computationally efficient method. On the other hand, ignoring technical issues, the stabilization motivated by $\mathbf{\Delta t \Lambda^* \Lambda}\left(\mathbf{u}^{n+1}-\mathbf{u}^{n-1}\right)$ *which is most natural in appearance is to include only a boundary integral term in both equations of the forms*

$$
\Delta t g^2\int_I(\phi_h^{n+1}-\phi_h^{n-1})\psi_h ds \text{ and } \Delta t\int_I\left(u_h^{n+1}-u_h^{n-1}\right)\cdot\widehat{n}v_h\cdot\widehat{n}ds.
$$

*It is an open problem to analyze if this stabilization suffices. The inequality (*TRACE*) above suggests that the stabilization in (*SDSTAB*) is closely connected.*

Theorem 3 (Unconditional stability of (SDstab)**).** (SDstab) *is stable: for any $N>0$, there holds*

$$\begin{aligned}&\frac{1}{2}\left[\|u_h^{N+1}\|_{DIV}^2+\|u_h^N\|_{DIV}^2\right]+gS_0\left(\|\phi_h^{N+1}\|_p^2+\|\phi_h^N\|_p^2\right)\\&+\Delta t\sum_{n=1}^{N}\left[\nu\|\nabla\left(u_h^{n+1}+u_h^{n-1}\right)\|_f^2+gk_{\min}\|\nabla\left(\phi_h^{n+1}+\phi_h^{n-1}\right)\|_p^2\right]\\&\le\|u_h^1\|_{DIV}^2+\|u_h^0\|_{DIV}^2+gS_0\|\phi_h^1\|_p^2+2\Delta t^2g^2\|\phi_h^1\|_{H^1(\Omega_p)}^2\\&+gS_0\|\phi_h^0\|_p^2+2\Delta t^2g^2\|\phi_h^0\|_{H^1(\Omega_p)}^2+2\Delta t\left[c_I(u_h^1,\phi_h^0)-c_I(u_h^0,\phi_h^1)\right]\\&+2\Delta t\sum_{n=1}^{N}\left[(f_f^n,u_h^{n+1}+u_h^{n-1})_f+g(f_p^n,\phi_h^{n+1}+\phi_h^{n-1})_p\right].\end{aligned}$$

Proof. We adapt the proof of Theorem 1 to the setting of (SDstab). Set $v_h=2\Delta t(u_h^{n+1}+u_h^{n-1})$, $\psi_h=2\Delta t(\phi_h^{n+1}+\phi_h^{n-1})$, then add and subtract $||u_h^n||_{DIV}^2$ and $gS_0\|\phi_h^n\|_p^2+2\Delta t^2g^2\|\phi_h^n\|_{H^1(\Omega_p)}^2$. This gives the total energy estimate as

$$\begin{aligned}&E^{n+1/2}-E^{n-1/2}+2\Delta t\left(C^{n+1/2}-C^{n-1/2}\right)+\Delta tD^{n+1/2}\\&=2\Delta t\left(\,(f_f^n,u_h^{n+1}+u_h^{n-1})_f+g(f_p^n,\phi_h^{n+1}+\phi_h^{n-1})\right).\end{aligned}\tag{14}$$

Here

$$\begin{aligned}E^{n+1/2}&=||u_h^{n+1}||_{DIV}^2+||u_h^n||_{DIV}^2\\&+gS_0\left(||\phi_h^{n+1}||_p^2+||\phi_h^n||_p^2\right)+2\Delta t^2g^2\left(||\phi_h^{n+1}||_{H^1(\Omega_p)}^2+||\phi_h^n||_{H^1(\Omega_p)}^2\right),\\D^{n+1/2}&=a_f(u_h^{n+1}+u_h^{n-1},u_h^{n+1}+u_h^{n-1})+a_p(\phi_h^{n+1}+\phi_h^{n-1},\phi_h^{n+1}+\phi_h^{n-1}),\\C^{n+1/2}&=c_I(u_h^{n+1},\phi_h^n)-c_I(u_h^n,\phi_h^{n+1}).\end{aligned}$$

Standard coercivity estimates show that

$$D^{n+1/2}\ge\nu||\nabla\left(u_h^{n+1}+u_h^{n-1}\right)||_f^2+gk_{min}||\nabla\left(\phi_h^{n+1}+\phi_h^{n-1}\right)||_p^2.$$

Sum (14) from $n=1,\dots,N$ and stability and the stated energy inequality follows provided

$$E^{N+1/2}+2\Delta tC^{N+1/2}\ge\frac{1}{2}\left[\|u_h^{N+1}\|_{DIV}^2+\|u_h^N\|_{DIV}^2\right]+gS_0\left(\|\phi_h^{N+1}\|_p^2+\|\phi_h^N\|_p^2\right).$$

Consider the coupling terms. Using (trace)

$$\begin{aligned}2\Delta tC^{N+1/2}&=2\Delta tg\int_I\phi_h^Nu_h^{N+1}\cdot\widehat{n}_f-\phi_h^{N+1}u_h^N\cdot\widehat{n}_fds\\&\le2\Delta tg\left(||u_h^{N+1}||_{DIV}||\phi_h^N||_{H^1(\Omega_p)}+||u_h^N||_{DIV}||\phi_h^{N+1}||_{H^1(\Omega_p)}\right)\\&\le\frac{1}{2}\left[||u_h^{N+1}||_{DIV}^2+||u_h^N||_{DIV}^2\right]+2\Delta t^2g^2\left[||\phi_h^{N+1}||_{H^1(\Omega_p)}^2+||\phi_h^N||_{H^1(\Omega_p)}^2\right].\end{aligned}$$

Subtract this from $E^{N+1/2}$, cancel terms and complete the proof to find, that indeed

$$E^{N+1/2}+2\Delta t C^{N+1/2} \geq \frac{1}{2}\left[\|u_h^{N+1}\|_{DIV}^2 + \|u_h^N\|_{DIV}^2\right]+gS_0\left(\|\phi_h^{N+1}\|_p^2 + \|\phi_h^N\|_p^2\right) > 0.$$

■

3.2. Application to Stokes flow plus a Coriolis force term

The use of (CNLF) in geophysical flows is based on fast-slow wave decompositions and time filters, see [3], [4]. There are many complexities in geophysics we shall avoid in this application by focusing on stability of (CNLFSTAB) for Stokes flow plus strong rotation given by a Coriolis force term $f_C \times u$:

$$\begin{gathered} u_t - \nu\Delta u + \nabla p + f_C \times u = f(x,t), \text{ and } \nabla\cdot u = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \text{ and } u(x,0) = u_0(x) \text{ in } \Omega. \end{gathered}$$

Choose conforming velocity-pressure FEM spaces X^h, Q^h satisfying the usual discrete inf-sup / LBB^h condition for stability of the discrete pressure, e.g., [33], [34], [35],

$$X^h \subset X := \left(H_0^1(\Omega)\right)^d, \; Q^h \subset L_0^2(\Omega).$$

We denote the usual $L^2(\Omega)$ norm and the inner product by $\|\cdot\|$ and $(\cdot,\cdot)$. Let $\Lambda u := f_C \times u$. The (CNLFSTAB) realization is then: find $(u_h^{n+1}, p_h^{n+1}) \in X^h \times Q^h$ satisfying, for all $v_h \in X^h, q_h \in Q^h$,

$$\begin{aligned} &\left(\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}, v_h\right) + \Delta t\left(\Lambda(u_h^{n+1}-u_h^{n-1}), \Lambda v_h\right) \\ &+(\Lambda u_h^n, v_h) + \nu\left(\frac{\nabla u_h^{n+1}+\nabla u_h^{n-1}}{2}, \nabla v_h\right) \qquad \text{(SCSTAB)} \\ &-\left(\frac{p_h^{n+1}+p_h^{n-1}}{2}, \nabla\cdot v_h\right) + \left(q_h, \nabla\cdot\left(\frac{u_h^{n+1}+u_h^{n-1}}{2}\right)\right) = (f^n, v_h). \end{aligned}$$

Theorem 4. *(Unconditional Stability).* (SCSTAB) *is unconditionally stable. Specifically, for any $N > 0$, the energy estimate holds*

$$\begin{aligned} &\tfrac{1}{2}\|u^{N+1}\|^2 + \|u^N\|^2 + 2\Delta t^2\|\Lambda u^{N+1}\|^2 + \frac{\Delta t\nu}{2}\sum_{n=1}^{N}\|\nabla(u_h^{n+1}+u_h^{n-1})\|^2 \\ &\leq \|u_h^1\|^2 + \|u_h^0\|^2 + 2\Delta t^2(\|\Lambda u_h^1\|^2 + \|\Lambda_h^0\|^2) + 2\Delta t(\Lambda u_h^0, u_h^1) + \frac{2\Delta t}{\nu}\sum_{n=1}^{N}\|f^n\|_*^2. \end{aligned} \tag{15}$$

Proof. In (SCSTAB) set $v_h = u_h^{n+1} + u_h^{n-1}$ and multiply through by $2\Delta t$. Using the same notation as in Theorem 1, this gives, after adding and subtracting $\|u_h^n\|^2 + 2\Delta t^2 \|\Lambda u_h^n\|^2$,

$$E^{n+1/2} - E^{n-1/2} + 2\Delta t(C^{n+1/2} - C^{n-1/2}) \\ + \Delta t\nu \|\nabla(u_h^{n+1} + u_h^{n-1})\|^2 = 2\Delta t(f^n, u_h^{n+1} + u_h^{n-1}).$$

Applying Young's inequality to the RHS, the above reduces to

$$E^{n+1/2} - E^{n-1/2} + 2\Delta t(C^{n+1/2} - C^{n-1/2}) \\ + \frac{\Delta t\nu}{2} \|\nabla(u_h^{n+1} + u_h^{n-1})\|^2 \leq \frac{2\Delta t}{\nu} \|f^n\|_*^2,$$

where $\|f^n\|_* = \sup_{v_h \neq 0, v_h \in X^h} (f^n, v_h)/\|\nabla v_h\|$.

Sum from $n = 1$ to N to obtain

$$E^{N+1/2} + 2\Delta t C^{N+1/2} + \tfrac{\Delta t\nu}{2} \sum_{n=1}^{N} \|\nabla(u_h^{n+1} + u_h^{n-1})\|^2 \leq E^{1/2} + 2\Delta t C^{1/2} + \tfrac{2\Delta t}{\nu} \sum_{n=1}^{N} \|f^n\|_*^2.$$

By the Cauchy-Schwarz inequality, we have

$$E^{N+1/2} + 2\Delta t C^{N+1/2} \geq \tfrac{1}{2} ||u^{N+1}||^2 + ||u^N||^2 + 2\Delta t^2 ||\Lambda u^{N+1}||^2,$$

and the stability inequality (15) follows. ■

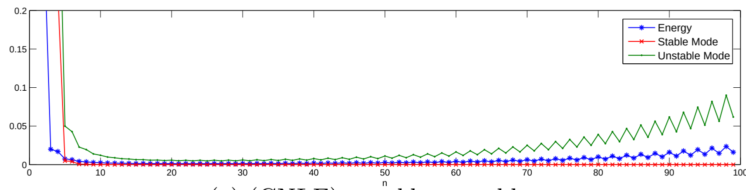
4. Numerical Illustrations

We present two tests of stability, one for Stokes-Darcy and one for Stokes flow plus strong rotation.

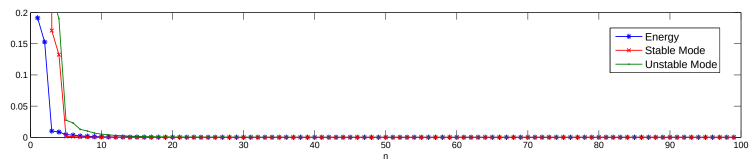
Example 1: Stokes-Darcy. We solve the Stokes-Darcy problem, with and without stabilization, for small values of the parameter S_0 (all other parameters are set to 1.0) for $T_{final} = 10$. Tests 1 and 2 use the exact solutions (see (16)) introduced by Mu and Zhu in [24] satisfying the coupling conditions over the subdomains $\Omega_f = (0,1) \times (1,2)$ and $\Omega_p = (0,1) \times (0,1)$. We use Taylor-Hood elements (P2-P1) for the Stokes problem and piecewise quadratics (P2) for the Darcy problem. The initial condition and first two terms are chosen to correspond with the exact solutions.

$$\begin{aligned} u(x,y,t) &= ((x^2(y-1)^2 + y)\cos(t), (\frac{2}{3}x(1-y)^3 + 2 - \pi\sin(\pi x))\cos(t)) \\ p(x,y,t) &= (2 - \pi\sin(\pi x))\sin(\frac{\pi}{2}y)\cos(t) \\ \phi(x,y,t) &= (2 - \pi\sin(\pi x))(1 - y - \cos(\pi y))\cos(t) \end{aligned} \tag{16}$$

Forcing terms are set to zero so that the true solution decays to zero rapidly as $t \to \infty$. Thus any growth in the approximate solution implies instability. All



(a) (CNLF): weakly unstable



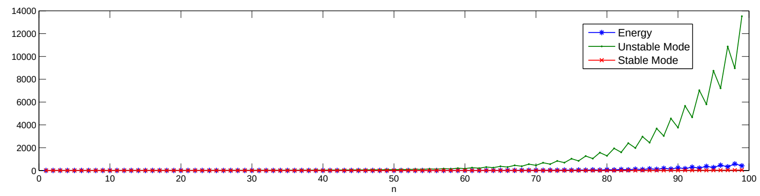
(b) (SDSTAB): unconditionally stable

Figure 1: Stokes-Darcy Test 1: $S_0 = 1.0$

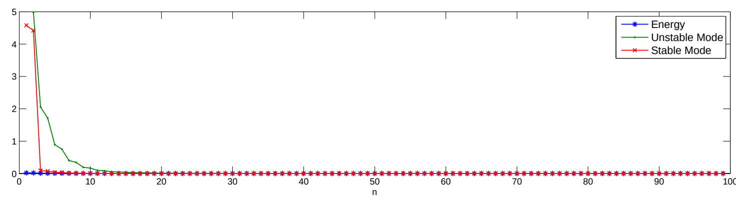
tests here were performed using FreeFEM++ [38]. Setting $h = \Delta t = 0.1$ violates the (CFL) condition for stability of the (CNLF) for Stokes-Darcy method given in [5]. In Test 1, $S_0 = 1.0$. (SDSTAB) is stable (Figure 1b) and both modes converge to zero as predicted, while after a long enough time, (CNLF) becomes weakly unstable (Figure 1a), as expected. In Test 2, $S_0 = 10^{-4}$. As predicted, (SDSTAB) is stable (Figure 2b) while spurious oscillations in the unstable mode correspond to an increase in system energy of (CNLF) making it unstable (Figure 2a).

We have preformed tests for other parameter values with the same results: (CNLF) becomes unstable if (CFL) is violated, sometimes weakly, and sometimes drastically, while (SDSTAB) remains stable, as predicted in Theorems 1 and 2.

Example 2: Stokes flow + strong rotation. In this example we consider the $2d$ Stokes problem plus Coriolis forces with a speed of rotation $\omega = 100$. The computational domain is the square $[0,1] \times [0,1]$. Let $g_1(x) = x^2(1-x^2)\exp(7x)$, $g_2(y) = y^2(1-y)^2$ and define the initial condition by $u_0 = (g_1(x)g_2'(y), -g_1'(x)g_2(y))$. We solve the problem and plot the kinetic energy vs time for (CNLF) first without and then with stabilization. As predicted by the theory, (CNLF) is unstable until $\Delta t||\Lambda|| < 1$ (see Figure 3a), whereas (CNLFSTAB) remains stable for all time steps, as shown in Figure 3b.



(a) (CNLF): growth in unstable mode



(b) (SDSTAB): unconditionally stable

Figure 2: Stokes-Darcy Test 2: $S_0 = 10^{-4}$

5. Conclusions

The accepted view, e.g., [19], [21], of (CNLF) without additional stabilizations or time filters is that it has two issues: the CFL condition and the growth in the unstable mode. Time filters are a wonderfully elegant and modular tool that damps the unstable mode. Analysis of (CNLF) with the addition of the Robert-Asselin time filter (CNLF-RA) in [20] shows that stability of (CNLF) + time filters remains subject to CFL-type conditions.

We have presented a stabilization that, while not modular, eliminates the CFL condition and controls the unstable mode. Naturally, when a CFL condition is grossly violated (as the stability tests here did purposefully), the difficulty could be shifted from stability to accuracy. Thus, the next important step in studying (CNLFSTAB) must be precise error analysis and careful testing of accuracy for specific applications, like the two in Section 3.

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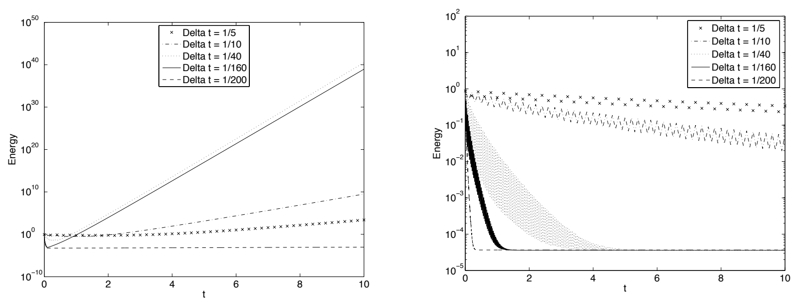

(a) (CNLF): growth in unstable mode (b) (SCSTAB): unconditionally stable

Figure 3: Stokes + Coriolis Force Test

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