Problem 1. Let $\sigma > 0$. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of functions $f_k : \mathbb{R} \to \mathbb{R}$ with $f_k(0) = 0$. Moreover let $(A_k)_{k \in \mathbb{N}} \subseteq [0, \infty)$ be a bounded sequence of real numbers such that
\[ |f_k(x) - f_k(y)| \leq A_k|x - y|^\sigma \quad \text{for all } x, y \in \mathbb{R}. \]

(a) Show that there exists $f : \mathbb{R} \to \mathbb{R}$ such that a subsequence $f_{k_i}$ converges uniformly to $f$ in every interval $[-a, a]$, $a > 0$.
(b) Show that $f$ satisfies
\[ |f(x) - f(y)| \leq A|x - y|^\sigma \]
where $A = \lim \inf_{k \to \infty} A_k$.

Problem 2. Prove that if $X$ is a metric space and $f : X \times [0, 1] \to \mathbb{R}$ is a continuous function, then $g : X \to \mathbb{R}$, defined by $g(x) = \sup_{t \in [0,1]} f(x, t)$, is continuous.

Problem 3. Prove (using only the material covered in the course) that there is no continuous and one-to-one function $f : \mathbb{R}^2 \to \mathbb{R}$. **Hint**: Assume that such a function exists and then restrict the function to the unit circle in $\mathbb{R}^2$.

Problem 4. Suppose $f : \mathbb{R}_+^2 \to \mathbb{R}$ is a continuous function defined on
\[ \mathbb{R}_+^2 = \{(x, y) : x \in \mathbb{R}, y > 0\}. \]
Assume also that the limits
\[ g(u, v) = \lim_{t \to 0^+} \frac{f((u + t) \cos v, (u + t) \sin v) - f(u \cos v, u \sin v)}{t}, \]
and
\[ h(u, v) = \lim_{t \to 0^+} \frac{f(u \cos(v + t)), u \sin(v + t)) - f(u \cos v, u \sin v)}{t}, \]
eexist and define continuous functions $g, h$ on the domain
\[ D = \{(u, v) : u > 0, \ 0 < v < \pi\}. \]
Prove that the function $f$ is differentiable on $\mathbb{R}_+^2$.

Problem 5. Let $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$, where $\Omega \subseteq \mathbb{R}^n$ is open and bounded. Let $\Delta f = \sum_{i=1}^n \partial^2 f / \partial x_i^2$ be the Laplace operator.

(a) Show that if for some $\varepsilon > 0$ and $x_0 \in \Omega$ we have $\Delta f(x_0) \geq \varepsilon$, then $f$ has no local maximum at $x_0$.
(b) Conclude that if $\Delta f(x) \geq \varepsilon$ for some $\varepsilon > 0$ and all $x \in \Omega$, then we have sup$_\Omega f = sup_{\partial \Omega} f$.
(c) Conclude that if $\Delta f(x) \geq 0$ for all $x \in \Omega$, then we have sup$_\Omega f = sup_{\partial \Omega} f$.

**Hint for part (c):** Observe that $\Delta |x|^2 = 2n$. Use it to modify a function $f$ in (c) so that you can apply part (b).

Problem 6. For $x = (x_1, x_2) \in \mathbb{R}^2$, let $|x| = \sqrt{x_1^2 + x_2^2}$. Let $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ and let $f : \overline{D} \to \mathbb{R}$ be continuous on $\overline{D}$. Prove that
\[ \lim_{n \to \infty} \int_D (n + 2)|x|^n f(x) \, dA = \int_0^{2\pi} f(\cos t, \sin t) \, dt. \]