Problem 1. Suppose that $a_{ij} \leq 1$ for all $i, j \in \mathbb{N}$. Suppose also that
\[
\forall i \ a_{i1} \leq a_{i2} \leq a_{i3} \leq \ldots \quad \text{and} \quad \forall j \ a_{1j} \leq a_{2j} \leq a_{3j} \leq \ldots
\]
Prove that the following iterated limits exist and they are equal
\[
\lim_{i \to \infty} \lim_{j \to \infty} a_{ij} = \lim_{j \to \infty} \lim_{i \to \infty} a_{ij}.
\]

Problem 2. Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix.

(a) Show that there exists $\Lambda > 0$ such that $|x| \leq \Lambda |Ax|$ for all $x \in \mathbb{R}^n$.

(b) Let $0 \leq \lambda < \Lambda^{-1}$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with $|f(x) - f(y)| \leq \lambda |x - y|$ for all $x, y \in \mathbb{R}^n$. Show that for any $y \in \mathbb{R}^n$, there exists exactly one $x \in \mathbb{R}^n$ satisfying $Ax + f(x) = y$.

Problem 3. Let $n \geq 2$.

(a) Show that there is a non-constant map $F : \mathbb{R}^n \to \mathbb{R}^n$ of class $C^\infty$ such that

\[
(F(x) - F(y), x - y) = 0 \quad \text{for all } x, y \in \mathbb{R}^n.
\]

(b) Show that if $f \in C^\infty(\mathbb{R}^n)$ and $F = \nabla f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1), then $F$ is constant.

Hint: To prove (b), use the definition of the directional derivative, apply it to $F = \nabla f$ and use (1) to obtain some identity for $D^2 f$. Then prove that $D^2 f = 0$ and conclude (with a short proof) that $F = \nabla f$ is constant. At some point you might need some facts from linear algebra about symmetric matrices.

Problem 4. Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded equicontinuous sequence of real-valued functions on a compact metric space $(X, d)$.

(a) Prove that the family $g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}$, $n \in \mathbb{N}$ is uniformly bounded and equicontinuous.

(b) Prove that the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly on $X$.

Problem 5. For $a = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$, $a_n \neq 0$ let $P_a(x) = a_n x^n + \ldots + a_1 x + a_0$. Suppose that for $a^o = (a_0^o, \ldots, a_n^o)$, $a_n^o \neq 0$ the polynomial $P_{a^o}(x)$ has $n$ distinct real roots. Prove that there is $\varepsilon > 0$ and $C^\infty$ smooth functions
\[
\lambda_1, \ldots, \lambda_n : B^{n+1}(a^o, \varepsilon) \to \mathbb{R}
\]
such that for any $a \in B^{n+1}(a^o, \varepsilon)$, $\lambda_1(a), \ldots, \lambda_n(a)$ are distinct roots of the polynomial $P_a(x)$. In other words, prove that in a small neighborhood of $a^o$, roots of the polynomial $P_a$ depend smoothly on the coefficients $a_0, a_1, \ldots, a_n$.

Problem 6. Let $\mathbb{T} := [0, 1]^2$ and assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\mathbb{Z}^2$-periodic, that is $f(x + k) = f(x)$ for all $x \in \mathbb{R}^2$ and all $k \in \mathbb{Z}^2$.

Prove that
\[
\int_{\mathbb{T}} f(x) \, dx = \int_{\mathbb{T}} f(x + z) \, dx \quad \text{for any } z \in \mathbb{R}^2.
\]