Problem 1. For $n$ a positive integer, put:

$$t_n = \frac{1}{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} - \frac{1}{2n+4} + \cdots + \frac{1}{4n-1} - \frac{1}{4n},$$

(2n terms in the right-hand-side).

Find, with proof, the following limit $T$:

$$T = \lim_{n \to \infty} nt_n.$$  

Hint: Relate the given limit to suitable Riemann sums for the function $(1 + x)^{-2}$.

Problem 2. Prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then

$$\lim_{n \to \infty} n \sqrt{\int_{a}^{b} |f(x)|^n \, dx} = \sup_{x \in [a, b]} |f(x)|.$$  

Problem 3. Let $\mathcal{M}$ denote the space of all real $2 \times 2$-matrices, equipped with the norm $||A|| = \sqrt{\text{tr}(A^T A)}$, for $A \in \mathcal{M}$ (here $A^T$ denotes the transpose of the matrix $A$ and for any $2 \times 2$ real matrix $B$, $\text{tr}(B)$ denotes its trace). Consider the map $F$ from $\mathbb{R}^2$ to $\mathcal{M}$ given by the formula, for any $(s, t) \in \mathbb{R}^2$:

$$F(s, t) = \begin{pmatrix} \cos(t) + \cos(s) & \sin(t) + \sin(s) \\ -\sin(t) + \sin(s) & \cos(t) - \cos(s) \end{pmatrix}.$$  

Denote by $\mathcal{N} \subset \mathcal{M}$ the space of all real $2 \times 2$ matrices of rank one and norm one.

Prove that the image of the map $F$ is the space $\mathcal{N}$ and that the map $F$ is a local homeomorphism to its image (the latter with the induced topology).

Problem 4. Let $\mathcal{F} \subset C^\infty[0, 1]$ be a uniformly bounded and equicontinuous family of smooth functions on $[0, 1]$ such that $f' \in \mathcal{F}$ whenever $f \in \mathcal{F}$. Suppose that

$$\sup_{x \in [0,1]} |f'(x) - g'(x)| \leq (1/2) \sup_{x \in [0,1]} |f(x) - g(x)|$$

for all $f, g \in \mathcal{F}$.

Show that there exists a sequence $f_n$ of functions in $\mathcal{F}$ that tends uniformly to $Ce^x$, for some real constant $C$.

Hint: Use the contraction principle. In order to apply the contraction principle you can use, without proof, the fact that if $X$ is a complete metric space, $A \subset X$ and $T : A \to X$ is uniformly continuous, then $T$ uniquely extends to a continuous map $\overline{T} : \overline{A} \to X$ defined on the closure $\overline{A}$.

Problem 5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping of class $C^1$. Prove that there is an open and dense set $\Omega \subset \mathbb{R}^n$ such that the function $R(x) = \text{rank} Df(x)$ is locally constant on $\Omega$, i.e. it is constant in a neighborhood of every point $x \in \Omega$.

Problem 6. Let $\Phi : \mathbb{R}^2 \to \Phi(\mathbb{R}^2) \subset \mathbb{R}^2$ be a diffeomorphism. Prove that

$$\int_{B^2(0,1)} \|D\Phi\| = \int_{\Phi(B^2(0,1))} \|D(\Phi^{-1})\|,$$

where $\|A\| = (\sum_{i,j=1}^2 a_{ij}^2)^{1/2}$ is the Hilbert-Schmidt norm of the matrix.

Hint. Compare $\|A\|$ and $\|A^{-1}\|$ for a $2 \times 2$ matrix.