Preliminary Exam August 2018

Problem 1. For n a positive integer, put:

$$t_n = \frac{1}{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} - \frac{1}{2n+4} + \cdots + \frac{1}{4n-1} - \frac{1}{4n},$$

(2n terms in the right-hand-side).

Find, with proof, the following limit $\mathcal{T}$:

$$\mathcal{T} = \lim_{n\to\infty} nt_n.$$

Hint: *Relate the given limit to suitable Riemann sums for the function* $(1+x)^{-2}$.

Problem 2. Prove that if $f : [a,b] \to \mathbb{R}$ is continuous, then

$$\lim_{n\to\infty} \sqrt[n]{\int_a^b |f(x)|^n \, dx} = \sup_{x\in[a,b]} |f(x)|.$$

Problem 3 Let $\mathcal{M}$ denote the space of all real 2×2-matrices, equipped with the norm $||A|| = \sqrt{\operatorname{tr}(A^T A)}$, for $A \in \mathcal{M}$ (here A^T denotes the transpose of the matrix A and for any 2×2 real matrix B, $\operatorname{tr}(B)$ denotes its trace). Consider the map F from $\mathbb{R}^2$ to $\mathcal{M}$ given by the formula, for any $(s,t) \in \mathbb{R}^2$:

$$F(s,t) = \ (1/2) \begin{vmatrix} \cos(t) + \cos(s) & \sin(t) + \sin(s) \\ -\sin(t) + \sin(s) & \cos(t) - \cos(s) \end{vmatrix}.$$

Denote by $\mathcal{N} \subset \mathcal{M}$ the space of all real 2×2 matrices of rank one and norm one.

Prove that the image of the map F is the space $\mathcal{N}$ and that the map F is a local homeomorphism to its image (the latter with the induced topology).

Problem 4. Let $\mathcal{F} \subset C^\infty[0,1]$ be a uniformly bounded and equicontinuous family of smooth functions on $[0,1]$ such that $f' \in \mathcal{F}$ whenever $f \in \mathcal{F}$. Suppose that

$$\sup_{x\in[0,1]} |f'(x) - g'(x)| \le (1/2) \sup_{x\in[0,1]} |f(x) - g(x)| \quad \text{for all } f, g \in \mathcal{F}.$$

Show that there exists a sequence f_n of functions in $\mathcal{F}$ that tends uniformly to Ce^x, for some real constant C.

Hint: *Use the contraction principle. In order to apply the contraction principle you can use, without proof, the fact that if X is a complete metric space, $A \subset X$ and $T : A \to X$ is uniformly continuous, then T uniquely extends to a continuous map $\overline{T} : \overline{A} \to X$ defined on the closure $\overline{A}$.*

Problem 5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping of class C^1. Prove that there is an open and dense set $\Omega \subset \mathbb{R}^n$ such that the function $R(x) = \operatorname{rank} Df(x)$ is locally constant on Ω, i.e. it is constant in a neighborhood of every point $x \in \Omega$.

Problem 6. Let $\Phi : \mathbb{R}^2 \to \Phi(\mathbb{R}^2) \subset \mathbb{R}^2$ be a diffeomorphism. Prove that

$$\int_{B^2(0,1)} \|D\Phi\| = \int_{\Phi(B^2(0,1))} \|D(\Phi^{-1})\|,$$

where $\|A\| = (\sum_{i,j=1}^2 a_{ij}^2)^{1/2}$ is the Hilbert-Schmidt norm of the matrix.

Hint. *Compare $\|A\|$ and $\|A^{-1}\|$ for a 2×2 matrix.*