Preliminary Exam in Analysis, August 2021

Problem 1. *Consider a function $u \in C^2(\Omega)$, where Ω is an open set in $\mathbb{R}^2$. For $x \in \Omega$ and $r < \operatorname{dist}(x, \partial\Omega)$ let $S_r(x) = \{y \in \mathbb{R}^2 : |x-y| = r\}$ the 1-dimensional sphere centered at x with radius r. Consider the mean value of u on $S_r(x)$ as a function of the radius r*

$$\phi(r) := \fint_{S_r(x)} u(y)\, d\sigma_r(y) \equiv \frac{1}{2\pi r}\int_{S_r(x)} u(y) d\sigma_r(y).$$

(1) *Set $y = x + r\,z$ and show that*

$$\phi(r) = \fint_{S_1(0)} u(x+rz)\, d\sigma_1(z)$$

(2) *Compute $\phi'(r)$ and use the Divergence Theorem to show that $\phi'(r) \equiv 0$ whenever $\operatorname{div}(\nabla u) = \partial_{11}u + \partial_{22}u = 0$; i.e. if the function u is harmonic.*

(3) *Deduce that harmonic functions satisfy the mean value property*

$$u(x) = \fint_{S_r(x)} u(y)\, d\sigma_r(y)$$

for all $x \in \Omega$ and all $r < \operatorname{dist}(x, \partial\Omega)$.

Problem 2. *Let (M, d) be a metric space. Prove that $\hat{d}(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ is also a metric, and that it generates the same topology as the original metric d; that is, show that any open set for $\hat{d}$ is an open set for d and vice versa.*

Problem 3. *Show that*

$$K := \{f \in C^1((0,1)) \cap C^0([0,1]) : \quad f'(x) = |f(x)| \text{ and } |f(x)| \le 2 \quad \text{holds for all } x \in (0,1)\}$$

is a compact set when equipped with the metric

$$d(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)| + \sup_{x\in(0,1)} |f'(x) - g'(x)|$$

Problem 4. *Assume $f(x)$ and $g(x)$ are power series around $x_0 = 0$ both with positive radius of convergence, i.e.*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{and} \qquad g(x) = \sum_{k=0}^{\infty} b_k x^k.$$

Show that if there exists a sequence $x_k \neq 0$ with $x_k \to 0$ such that $f(x_k) = g(x_k)$ then $f(x) = g(x)$ in their interval of convergence.

Problem 5. *Let S be the subset of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$, $n \ge 2$, consisting of pairs of vectors (v_1, v_2) such that $\|v_1\| = \|v_2\| > 0$ and $v_1 \cdot v_2 = 0$. Prove that S is a C^∞-smooth submanifold of $\mathbb{R}^{2n}$ of some dimension $1 \le k \le 2n$. Find the dimension k of S.*

Problem 6. *A set $\Omega \subset \mathbb{R}^n$ is called star-shaped with respect to a point $x_0 \in \Omega$, if for each $x \in \Omega$ the segment connecting x to x_0 is contained in Ω. Prove that if $\Omega \subset \mathbb{R}^n$ is open and star-shaped with respect to some x_0, and $f \in C^2(\Omega)$ is such that*

$$v^t D^2 f(x) v \ge 0 \quad \forall v \in \mathbb{R}^n, \quad x \in \Omega,$$

then there exists $A \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$f(x) \ge \langle A, x\rangle + b \quad \text{for all } x \in \Omega.$$

Note that a star-shaped domain is not necessarily convex. You cannot use any result about convex functions without proving it.