Preliminary Exam in Analysis 4/29/14
Identification number:

20 points per question.
The best six questions will count.

Question 1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by the formulas:

$$f(0,0) = 0,$$

$$f(x,y) = \frac{(x+y)^3}{x^2+y^2}, \text{ for any } (x,y) \in \mathbb{R}^2, \text{ with } (x,y) \neq (0,0).$$

Prove that f is everywhere Lipschitz, but not everywhere differentiable.

Question 2

Prove the identity, valid for any real x, with $|x| < 1$:

$$\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k.$$

Now let $f : (-3,3) \to \mathbb{R}$ be given by the series, valid for any real x with $|x| < 3$:

$$f(x) = \sum_{k=1}^{\infty} \left(\frac{x}{(-1)^k + 4} \right)^k.$$

Prove that $|f'(x)| \leq \dfrac{3}{(3-x)^2}$, for any real number x, such that $0 \leq x < 3$.

Question 3

Show that $[0,1]$ can not be written as a countably infinite union of disjoint closed intervals.

Question 4

Let $f(x, y) = (x - y,\ xy)$, for any (x, y) in $\mathbb{R}^2$, with $x > 0$ and $y > 0$. Show that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective onto its image and has a smooth inverse. Also identify the image of f and determine the Jacobian matrix of the inverse function.

Question 5

Let $f_n(x) = e^{-nx}\left(1 + \dfrac{x}{n}\right)^{n^2}$, defined for any real x and for any $n \in \mathbb{N}$.

- Prove that there is a function $f : \mathbb{R} \to \mathbb{R}$, with $\lim_{n\to\infty} f_n(x) = f(x)$, for each real number x and determine this function explicitly.

- Is the convergence of $f_n(x)$ to $f(x)$ uniform? Discuss.

Question 6

Let $\mathcal{F} = \{f_n : \mathbb{R} \to \mathbb{R}; n \in \mathbb{N}\}$ be a sequence of $\mathcal{C}^1$ functions satisfying the conditions, for each $n \in \mathbb{N}$:

- $f_n(0) = 0$,

- $|f_n'(x)| < \dfrac{n^2 + x^4}{n^2 + x^2}$, for all $x \in \mathbb{R}$.

Show that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a subsequence $\mathcal{G} = \{f_{n_k} : k \in \mathbb{N}\}$ of $\mathcal{F}$, such that for each $x \in \mathbb{R}$:

$$\lim_{k\to\infty} f_{n_k}(x) = f(x).$$

Also prove that the convergence of the subsequence $\mathcal{G}$ to its limit is uniform on compact subsets of the reals.

Question 7

For n a positive integer, let $u : \mathbb{R}^n - \{0\} \to \mathbb{R}$ be a $\mathcal{C}^2$ function.
Suppose that u depends only on the variable $r = \sqrt{\underline{x}.\underline{x}}$ and that u is bounded on its domain.
Finally suppose also that u is harmonic: $\underline{\nabla}.\underline{\nabla}u = 0$.
Prove that u is constant.

Question 8

Let $(\mathbb{X}, d)$ be a metric space.
Denote by $(\mathcal{C}(\mathbb{X}), D)$ the metric space of continuous bounded real-valued functions on $\mathbb{X}$, equipped with the metric $D(f, g) = \sup_{x\in\mathbb{X}}(|f(x) - g(x)|)$, for any f and g in $\mathcal{C}(\mathbb{X})$.
Denote by a a fixed point of $\mathbb{X}$.
For each fixed $y \in \mathbb{X}$ define $f_y(x) = d(x, y) - d(x, a)$, for any $x \in \mathbb{X}$.
Prove that $f_y \in \mathcal{C}(\mathbb{X})$ and that the map $C : \mathbb{X} \to \mathcal{C}(\mathbb{X})$, $y \to C(y) = f_y$, defined for any $y \in \mathbb{X}$ is an isometry of $\mathbb{X}$ into $\mathcal{C}(\mathbb{X})$.
Is the image of the map C closed? Discuss.