An Experience in Undergraduate Research

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Undergraduate Mathematics Seminar
Order of topics

- How I got involved
- Research overview
- General chronological account
- Lessons learned upon analysis of experience
- Logistical considerations and advice
Introduction

- Research project took place over past summer
  - topic was discrete geometry
  - tangentially related to the Reinhardt Conjecture
Getting involved
Involvement

- At the time, I was a junior, contemplating graduate school
  - what makes a strong graduate application?
- Applied to REUs
  - not accepted
  - discussed outcome with a teacher at the time, Tom Hales
    - outcome of discussion was a two month project over the summer
Overview
How to pack the plane (with pentagons)

Arbitrary packing:

Optimal packing:

Packing density $= \frac{5 - \sqrt{5}}{2} \approx 0.9213$
Reinhardt Conjecture

Which convex, centrally symmetric shape has the worst optimal lattice packing?

It can’t be the pentagon both because the pentagon has a packing density which is strictly greater than the circle \((5\sqrt{5})/3 > \pi/\sqrt{12})\), and because it is not centrally symmetric.

The Reinhardt Conjecture hypothesizes the answer is the smoothed octagon, where the corners of the octagon have been replaced with hyperbolic arcs of maximal curvature.

It has been proven that the shape exists, has no corners, and is subject to a hexagonal symmetry constraint.
Dubins car problem

As stated by Hales in his ‘Bad Packings’ presentation:

“The Dubins problem is to take a car that is at a certain position and direction in the plane and navigate it to an ending position and direction in such a way that the absolute value of the curvature of the path is at most 1, and to find the shortest such path.”

This seemingly unrelated problem allows the Reinhardt conjecture to be re-contextualized as an optimal control problem.
Re-interpret Reinhardt

Smoothed octagon

Optimal control (Dubins) problem

Straight line segments and segments of maximal curvature in both situations
An optimal control problem

Because the Reinhardt conjecture can be re-contextualized as a problem in optimal control theory, it must possess the same properties as an optimal control problem.

According to Wikipedia:
“Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost function.”

Furthermore, the Reinhardt problem inherits another property of the Dubins car problem: it has a bang-bang solution.

Control simplex:

\[ U = \{(u_0, u_1, u_2) \in \mathbb{R}^3 | \sum_{i=0}^{2} u_i = 1, u_i \geq 0 \} \]

Differential equations:

\[ x' = f_1(x, y; u) := \frac{y(b+2ax-cx^2+cy^2)}{b+2ax-cx^2-cy^2} \]

\[ y' = f_2(x, y; u) := \frac{2(a-cx)y^2}{b+2ax-cx^2-cy^2} \]

Cost:

\[ \int_0^{t_f} \frac{x^2 + y^2 + 1}{y} \, dt \]

*a, b, and c are functions of \( u = (u_0, u_1, u_2) \)*
Simplifying

The solution is bang-bang; in other words, the control state can only be at one of three extremal points of $U$, which are the vertices of the triangle:

- $e_1 = (1, 0, 0)$
- $e_2 = (0, 1, 0)$
- $e_3 = (0, 0, 1)$

This drastically simplifies the differential equations, especially in the cases of $e_2$ and $e_3$. For $e_2$ and $e_3$, the solutions to these equations are **lines**. The solution for $e_1$ control is more complicated, but turns out to be **circular arcs**. The precise equations are dependent on the initial conditions.

All trajectories take place in the upper half-plane of the complex plane; specifically, on a constrained region.

A picture of the star region. The complete trajectory must lie in the interior of this region.
Correspondence with shape

Given a trajectory that is traced out in the star region given a particular set of controls, we can construct a shape in the traditional 2-d plane. The equation is complicated, so it is sufficient to recognize the existence of a correspondence, without explicitly detailing the correspondence. For completion’s sake, the relationship is:

\[ \gamma(\kappa, z_0, t) = \gamma_{k_1}(z_0, T_0, T_1, t)\gamma_{k_2}(z_1, T_1, T_2, t) \cdots \gamma_{k_n}(z_{n-1}, T_{n-1}, T_n, t) \]

where

\[ \kappa = ((k_1, t_1), (k_2, t_2), \ldots, (k_n, t_n)) \]

The result is a matrix in \( \text{SL}_2(\mathbb{R}) \). This matrix acts on the sixth roots of unity (n-th roots of unity for n=6), generating a hexagonally symmetric shape. Note that the shape is not necessarily closed.
To summarize

Given Reinhardt problem's corresponding differential equations for the optimal control, we can solve the differential equations assuming bang-bang control. These paths in the star region of the upper half-plane, when pieced together, correspond to a shape in the traditional 2-d plane. It is this shape that we are packing optimally in the plane, and the optimal packing density is the cost.
My research

- Given the properties of the optimal control problem as outlined by Hales, can we explicitly construct a closed shape whose cost is strictly less than that of the circle?

\[
\text{Cost} = \frac{\pi}{\sqrt{12}} \approx 0.906899682117109
\]
How to go about this

Via the process previously described, we can construct a circle with constant control \( \mathbf{u} = (1/3, 1/3, 1/3) \), where the solution is \( x + iy = 0 + i \cdot 1 = i \). Thus to construct a shape with strictly worse packing density, we would like to \textbf{deform the circle} in some hexagonally symmetric way.

We do this by setting our initial conditions in the star region at \( z = i \), then taking bang-bang control in some manner such that we return to \( z = i \) in finite time.

A desirable method of construction should:

a) be (reasonably) easy to compute
b) use bang-bang control
Minimize computation

Arbitrary control trajectory:

‘Triangle’ control trajectory:
Benefit of triangular control trajectory

- It depends on only one parameter: $x_1$
  - minimizes computation
Does it work?

Examples of triangular control trajectories:

Corresponding shapes:
It’s hard to tell

By visual inspection, it is not particularly obvious whether a given trajectory is going to form a shape that is a continuously differentiable deformation of the circle.

We developed a more systematic method to quantify whether a shape had the potential to be a smooth deformation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Error</th>
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<tbody>
<tr>
<td>[0, 0.00000000000000000000, 0]</td>
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</tbody>
</table>

As the parameter is increased, the error grows larger.
Other ideas?

We needn't restrict ourselves to this ‘triangle’ trajectory. Rather, we can develop new trajectories. The game becomes how to piece together solutions of the differential control equations, that is, lines and circular arcs, such that we start and end at $z = i$.

In principle, the way we go about this task is arbitrary without any fundamental insight; however, we would still like to minimize computation to the extent that we can.

Thus, we developed these trajectories, perhaps considered to resemble an hourglass and a cloverleaf. These have 2 and 3 parameters, respectively. More parameters yields more control.
Any luck?

Two, three parameter control trajectories

Corresponding shapes
What do the computations say?

While the shapes are nearly closed, we also require them to be **smooth deformations** of the unit circle, meaning that they are continuously differentiable. The prior examples are not. But could there be others.

This list of ‘errors’ through three-parameter trajectories of equal inputs illustrates how close we can come an error of 0.

```
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```
Unexpected outcome

- ‘Clover leaf’ captures three-piece hyperbolic symmetry of star region
  - yet we still find no solution

- What arrangement of controls will yield our desired outcome?
Further information

- reinhardt-control-trajectory
  - GitHub, matthew-gerstbrein
  - code and project synopsis

- “The Reinhardt Conjecture as an Optimal Control Problem”, Thomas Hales
  - on arXiv
Chronology
Beginning stages

- Introduction to project and relevant literature
  - Initial meetings with Dr. Hales to provide overview
    - Identify portions of paper that are and are not relevant
  - Take first steps in developing some technical details and begin building momentum
Early work

- Algebraic manipulations of equations
  - (pencil and paper work)
- Understanding geometry of the situation
  - many conversations to reach such a point
- Developing general intuition
Mid-End stage work

- Heavy emphasis on coding
  - equally heavy emphasis on debugging
- Aggregating results into cohesive results
  - organizing work helps
Typical research day(s)

- Various types of research tasks
  - reading, discussions, thinking, paperwork, coding, debugging
  - certain tasks were more heavily emphasized during certain periods than others
Recurring themes

- Non-linear progress
- (Sub) problems that need to be solved
  - not obvious how to solve them
- Trouble with code
- Combining different subjects in math
  - e.g. linear algebra, differential equations, (abstract) algebra, geometry, etc.
Lessons to take away

- There are many directions research can take
  - often the path forward is ambiguous
- Meetings with advisor are crucially important
- Entire process can potentially be a learning experience
Advice and Logistical Considerations

- Sufficient coursework
- Appropriate advisor
  - the person whom you work with is more important than the subject you work on
- Importance of subject preference
  - not as much as you’d think
- Time constraints
- If in doubt, weight answer towards ‘yes’
- Ask
If you choose to take on a project...

- Read, ask questions
- Strong communication with advisor
- Keep a ‘journal’
- Be consistent
- Treat it as what it should be: an enjoyable experience with the opportunity for personal growth
Thanks for your time!