Problem 1. Suppose $f: [1, \infty) \to \mathbb{R}$ is Riemann integrable on $[1, A]$ for every $A > 1$, and $\lim_{x \to \infty} x f(x) = 1$. Show that

$$\lim_{t \to \infty} \frac{1}{t} \int_1^e f(x) \, dx = 1$$

Problem 2.

(1) If every member of a sequence $\{f_n\}$ of functions on $[0, 1]$ is $K$-Lipschitz for a fixed $K > 0$ — that is, if $|f_n(y) - f_n(x)| \leq K|y - x|$ for all $x, y \in [0, 1]$ and $n \in \mathbb{N}$ — show that $\{f_n\}$ converges pointwise if and only if it converges uniformly.

(2) Prove that $\lim_{n \to \infty} \int_0^1 n \log \left(1 + \frac{x}{n}\right) \, dx = \frac{1}{2}$.

Problem 3. Show that the Hilbert cube

$$\mathcal{H} = \{(x_1, x_2, \ldots) | 0 \leq x_n \leq 2^{-n} \text{ for each } n \in \mathbb{N}\}$$

is compact when it is equipped with the metric $d(x, y) = \sup\{|x_n - y_n| \mid n \in \mathbb{N}\}$, where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. (Hint: use a diagonal argument.)

Problem 4. Prove that the series below converges uniformly on $[a, \infty)$ for any $a > 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{n^x}{1 + n^4 x^2}$$

Now for a square $N$ (i.e. with $\sqrt{N} \in \mathbb{N}$) show that

$$f \left(\frac{1}{N}\right) \geq \frac{N}{2} \sum_{n \geq \sqrt{N}} \frac{1}{n^3},$$

and using an integral to estimate the sum, show that $f \left(\frac{1}{N}\right) \geq 1/4$. Conclude that the series does not converge uniformly on $\mathbb{R}$.

Problem 5. For $n \geq 1$, suppose $f: \mathbb{R}^n \to \mathbb{R}$ is second-differentiable, and $(D^2f)_p$ is positive semidefinite for all $p$; i.e. $(D^2f)_p(v, v) \geq 0$ for all $v \in \mathbb{R}^n$. Show that $f$ is convex in the sense that for any $p, q \in \mathbb{R}^n$ and $t \in [0, 1]$,

$$f(p + t(q - p)) \leq f(p) + t(f(q) - f(p)) = (1 - t)f(p) + tf(q).$$

(In particular, you must show this for $n = 1$. Hint: in the general case, reduce to this one.)

Problem 6. Given $(p, q) \in \mathbb{R}^2$, let $f_{(p, q)}: \mathbb{R} \to \mathbb{R}$ be the polynomial:

$$f_{(p, q)}(x) = x^3 - px + q$$

Prove that for any $(y, z) \in \mathbb{R}^2$, there is a unique $(p, q) \in \mathbb{R}^2$ such that $f_{(p, q)}(y) = f_{(p, q)}(z) = 0$, and that the map $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ taking $(y, z)$ to this $(p, q)$ is smooth.

Prove that $\phi$ maps at most six-to-one, and that for $(y_0, z_0) \in \mathbb{R}^2$ with $\phi(y_0, z_0) = (p_0, q_0)$, $\phi$ restricts to a diffeomorphism from a suitable neighborhood of $(x_0, y_0)$ to a neighborhood of $(p_0, q_0)$ if and only if $\phi^{-1}(p_0, q_0)$ contains exactly six points.