PRELIMINARY EXAMINATION IN ANALYSIS
MAY 4, 2017

Problem 1. Suppose $f\colon [1,\infty) \to \mathbb{R}$ is Riemann integrable on $[1, A]$ for every $A > 1$, and $\lim_{x\to\infty} xf(x) = 1$. Show that

$$\lim_{t\to\infty} \frac{1}{t} \int_1^{e^t} f(x)\,dx = 1$$

Problem 2.

(1) If every member of a sequence $\{f_n\}$ of functions on $[0,1]$ is K-Lipschitz for a fixed $K > 0$ — that is, if $|f_n(y) - f_n(x)| \leq K|y - x|$ for all $x, y \in [0,1]$ and $n \in \mathbb{N}$ — show that $\{f_n\}$ converges pointwise if and only if it converges uniformly.

(2) Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 n \log\left(1 + \frac{x}{n}\right)\,dx = \frac{1}{2}$.

Problem 3. Show that the *Hilbert cube*

$$\mathcal{H} = \{(x_1, x_2, \ldots) \,|\, 0 \leq x_n \leq 2^{-n} \text{ for each } n \in \mathbb{N}\}$$

is compact when it is equipped with the metric $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_n - y_n| \,|\, n \in \mathbb{N}\}$, where $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$. (**Hint:** use a diagonal argument.)

Problem 4. Prove that the series below converges uniformly on $[a, \infty)$ for any $a > 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{nx}{1 + n^4 x^2}$$

Now for a square N (i.e. with $\sqrt{N} \in \mathbb{N}$) show that

$$f\left(\frac{1}{N}\right) \geq \frac{N}{2} \sum_{n \geq \sqrt{N}} \frac{1}{n^3},$$

and using an integral to estimate the sum, show that $f\left(\frac{1}{N}\right) \geq 1/4$. Conclude that the series does not converge uniformly on $\mathbb{R}$.

Problem 5. For $n \geq 1$, suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is second-differentiable, and $(D^2 f)_p$ is positive semidefinite for all p; i.e. $(D^2 f)_p(\mathbf{v}, \mathbf{v}) \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$. Show that f is *convex* in the sense that for any $p, q \in \mathbb{R}^n$ and $t \in [0,1]$,

$$f(p + t(q - p)) \leq f(p) + t(f(q) - f(p)) = (1 - t)f(p) + tf(q).$$

(In particular, you must show this for $n = 1$. **Hint:** in the general case, reduce to this one.)

Problem 6. Given $(p, q) \in \mathbb{R}^2$, let $f_{(p,q)}\colon \mathbb{R} \to \mathbb{R}$ be the polynomial:

$$f_{(p,q)}(x) = x^3 - px + q$$

Prove that for any $(y, z) \in \mathbb{R}^2$, there is a unique $(p, q) \in \mathbb{R}^2$ such that $f_{(p,q)}(y) = f_{(p,q)}(z) = 0$, and that the map $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ taking (y, z) to this (p, q) is smooth.

Prove that ϕ maps at most six-to-one, and that for $(y_0, z_0) \in \mathbb{R}^2$ with $\phi(y_0, z_0) = (p_0, q_0)$, ϕ restricts to a diffeomorphism from a suitable neighborhood of (x_0, y_0) to a neighborhood of (p_0, q_0) if and only if $\phi^{-1}(p_0, q_0)$ contains exactly six points.